EFFECT OF A RIGID ELLIPTIC DISK ON THE STRESS DISTRIBUTION IN AN ORTHOTROPIC PLATE

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By

ALVIN JEWEL OWENS

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TABLE OF CONTENTS

Chapter Page

I. INTRODUCTION 1

II. THE EXACT SOLUTION FOR PLANE STRESS 5

The Mathematical Formulation of the Boundary Conditions 6

The Derivation of the Differential Equation for an Orthotropic Plate in a State of Plane Stress 8

The General Solution of the Differential Equation 12

The Stresses Obtained from the Stress Function 15

The Exterior Boundary Conditions Satisfied 21

The Derivation of the Displacements u and v 22

A and B determined from the Interior Boundary Conditions 29

The Isotropic Stress Function as the Limiting Case of the Orthotropic Stress Function 33

III. THE EXACT SOLUTION FOR PLANE STRAIN 42

The Derivation of the Differential Equation for an Orthotropic Plate in a State of Plane Strain 42

A' and B' Determined from the Interior Boundary Conditions 44

TABLE OF CONTENTS--Continued

Chapter   Page

The Isotropic Stress Function as the Limiting Case of the Orthotropic Stress Function 49

IV. THE PLANE STRESS SOLUTION APPLIED TO A SITKA SPRUCE PLATE 52

APPENDIX 59

BIBLIOGRAPHY 82

LIST OF TABLES

Table Page

I. ELASTIC MODULI OF SITKA SPRUCE 60

II. CONSTANTS EVALUATED FOR SITKA SPRUCE. . . . 61

III. THE CONSTANTS A_1 AND B_1 EVALUATED FOR SITKA SPRUCE 62

IV. VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE x-AXIS EXTERIOR TO A RIGID CIRCULAR DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE 62

V. VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE y-AXIS EXTERIOR TO A RIGID CIRCULAR DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE 63

VI. VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE BOUNDARY OF A RIGID CIRCULAR DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE 63

VII. VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE LINE $x = 2a$ EXTERIOR TO A RIGID CIRCULAR DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE 64

LIST OF TABLES--*Continued*

Table Page

VIII. VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE x-AXIS EXTERIOR TO A RIGID ELLIPTIC DISK (b = 5a) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE 64

IX. VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE y-AXIS EXTERIOR TO A RIGID ELLIPTIC DISK (b = 5a) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE 65

X. VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE BOUNDARY OF A RIGID ELLIPTIC DISK (b = 5a) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE . . . . . [illegible]

XI. VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE x-AXIS EXTERIOR TO A RIGID ELLIPTIC DISK (a = 5b) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE . . . . [illegible]

XII. VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE BOUNDARY OF A RIGID ELLIPTIC DISK (a = 5b) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE . . . . . [illegible]

XIII. VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE BOUNDARY OF A RIGID ELLIPTIC DISK (a = 10b) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE . . . . . [illegible]

LIST OF FIGURES

Figure Page

1. Plain-sawn board relative to the log from which it is cut 2

2. Quarter-sawn board relative to the log from which it is cut 2

3. Elliptic Disk in an Orthotropic Plate. . . 7

4. Plain-sawn wooden plate with axes of elastic symmetry 53

5. Variation of the normal stress component X_x at points along the x-axis exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate . . . 68

6. Variation of the normal stress component Y_y at points along the x-axis exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate . . . 69

7. Variation of the normal stress component X_x at points along the y-axis exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate . . . 70

8. Variation of the normal stress component Y_y at points along the y-axis exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate . . . 71

LIST OF FIGURES--Continued

Figure Page

9. Variation of the shear stress component X_y at points along the boundary of a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate 72

10. Variation of the shear stress component X_y at points along the line x = 2a exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate 73

11. Variation of the normal stress component X_x at points along the x-axis exterior to a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce 74

12. Variation of the normal stress component Y_y at points along the x-axis exterior to a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce 75

13. Variation of the normal stress component X_x at points along the y-axis exterior to a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce 76

LIST OF FIGURES--Continued

Figure Page

14. Variation of the normal stress component X_y at points along the y-axis exterior to a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce 77

15. Variation of the shear stress component X_y at points along the boundary of a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce 78

16. Variation of the normal stress component X_x at points along the x-axis exterior to a rigid elliptic disk (a = 5b) with center at the origin for a plain-sawn plate of Sitka spruce 79

17. Variation of the shear stress component X_y at points along the boundary of a rigid elliptic disk (a = 5b) with center at the origin for a plain-sawn plate of Sitka spruce 80

18. Variation of the shear stress component X_y at points along the boundary of a rigid elliptic disk (a = 10b) with center at the origin for a plain-sawn plate of Sitka spruce 81

CHAPTER I

INTRODUCTION

A substance having three mutually perpendicular planes of elastic symmetry is said to be orthotropic. A thin plate of uniform thickness taken from such a material with its plane parallel to a plane of elastic symmetry will possess two perpendicular axes of elastic symmetry in the plane of the plate. Such a plate is said to be an orthotropic plate. In this dissertation, a large rectangular plate of this type with its edges parallel to the axes of symmetry and with a small rigid elliptic disk at its center is discussed. A uniform tension is assumed to act along two opposite edges of the plate and a mathematical analysis of the stress distribution near the elliptic disk is given. It is assumed that the strains are small and remain within the limits of perfect elasticity; that is, the material returns to its initial shape after the external load is removed.

It is well established that a plain-sawn or a quarter-sawn board can be considered as an orthotropic plate. The position of a plain-sawn board relative to the log from which it is cut is illustrated in Figure 1. Figure 2 illustrates the position of a quarter-sawn board relative to the log from

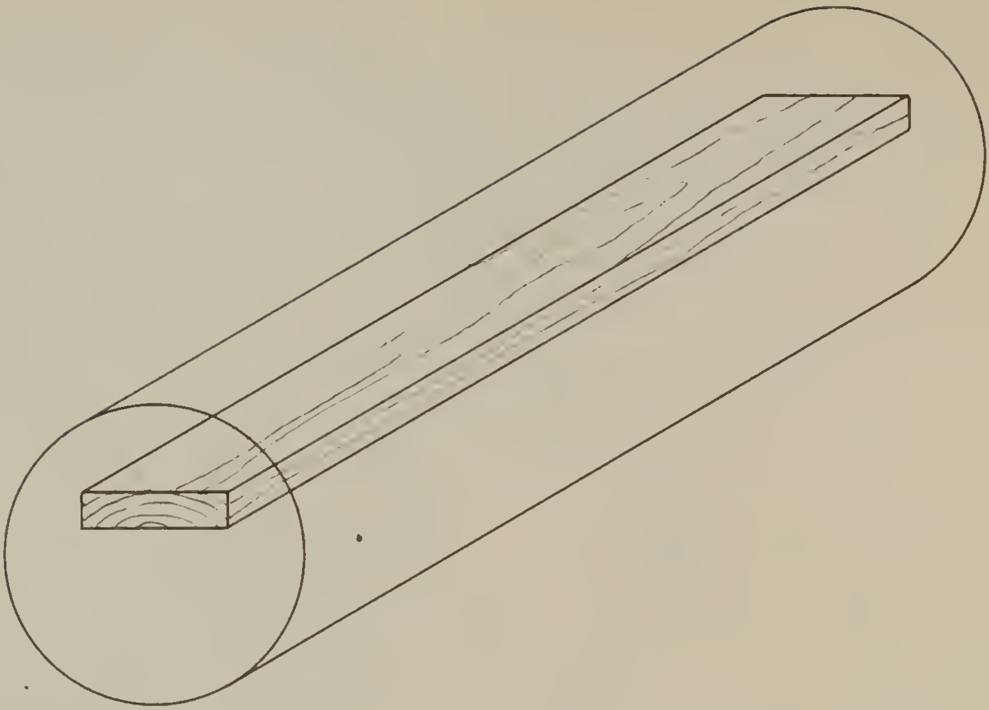

Figure 1.--Plain-sawn board relative to the log from which it is cut.

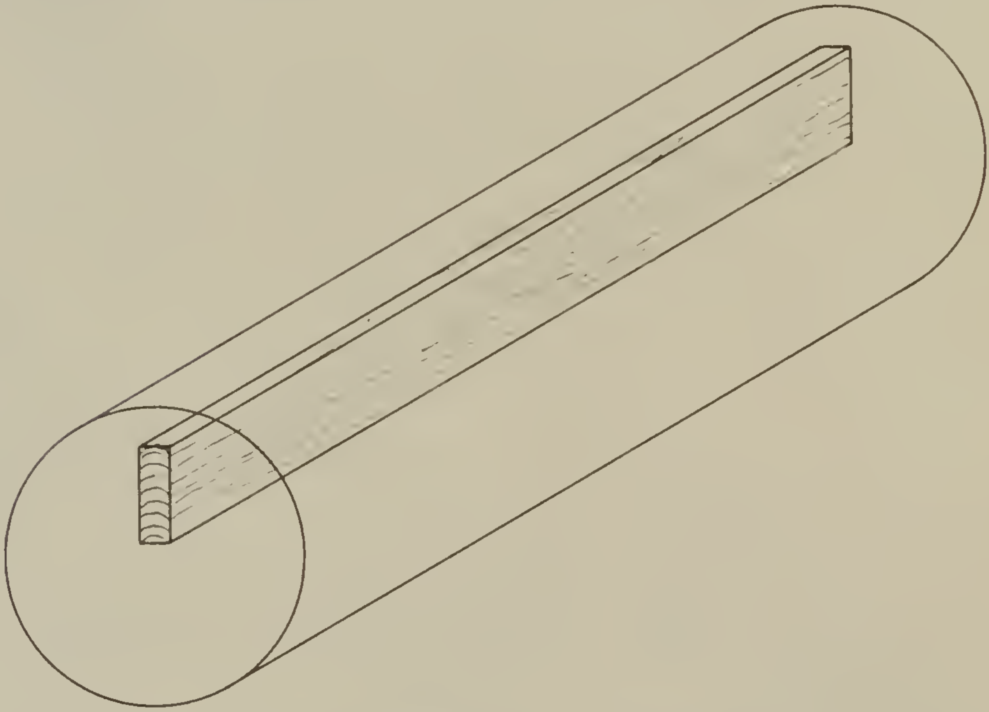

Figure 2.--Quarter-sawn board relative to the log from which it is cut.

which it is cut. The problem presented here is the mathematical idealization of the problem of a wooden plate pierced by an elliptic steel rod or, say, a wooden plate containing an elliptic knot.

It is desired to locate the points of stress concentration; that is, those points at which the ratio of the stress at the point to the normal stress at infinity is large compared to the ratio for other points in the plane. For a wooden plate, shearing stresses are probably most important in producing failure in the plate. Formulas for these ratios at any point on the boundary of and exterior to the rigid elliptic disk are obtained.

For the problem of a homogeneous orthotropic plate, in other words, a plate containing no holes or inserted disks, with the exterior boundary conditions given above the ratio of the shear stresses (in the directions perpendicular and parallel to the direction of applied tension) at any point to the normal stress at infinity equals zero. The ratio of the normal stress (in the direction perpendicular to the direction of the applied tension) at any point to the normal stress at infinity equals zero. The ratio of the normal stress (in the direction of the applied tension) at any point to the normal stress at infinity equals one. These ratios are constant for all points and there are no points of stress concentration.

A plate containing a hole or a rigid elliptic disk is non-homogeneous and points of stress concentration occur. Professor C. B. Smith[1] has solved the problem with the exterior boundary conditions given above for an infinite rectangular orthotropic plate containing at its center an elliptic hole. By taking the major and minor axes equal, the solution for a circular hole was obtained. For a plain-sawn plate of Sitka spruce containing a circular hole, a maximum tensile stress of 5.84S occurred on the edge of the hole at the ends of the diameter perpendicular to the direction of uniform tension. S is the uniform tension applied in the direction of the grain at two opposite edges. The maximum shear stress obtained was 0.71S.

The analysis of this dissertation applied to the stress distribution in a plain-sawn plate of Sitka spruce containing at its center a rigid circular disk gives a maximum tensile stress and a maximum sheer stress less than those found by Professor Smith. These results do not appear unreasonable. We would expect to weaken a wooden plate by drilling a hole in it. Also it seems reasonable that a close fitting steel disk in the hole would strengthen the plate, although the plate would be weaker than it was before the hole was drilled.

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[1] C. B. Smith, Effect of Elliptic or Circular Holes on the Stress Distribution in Plates of Wood or Plywood Considered as Orthotropic Materials, Forest Products Laboratory Report No. 1510, 1944.

This dissertation is divided into four chapters and an appendix. In Chapter II, the exact solution of the stress distribution in an infinite rectangular orthotropic plate containing at its center a rigid elliptic disk is given for the plate in a state of plane stress. In Chapter III, the exact solution of the stress distribution in an infinite rectangular orthotropic plate containing at its center a rigid elliptic disk is given for the plate in a state of plane strain. In both cases, the stress function obtained for the orthotropic plate is shown to contain as a limiting case the stress function for the isotropic plate. In Chapter IV, the solution of Chapter II is applied to a plain-sawn Sitka spruce plate. In the appendix tables of constants and tables of the stresses evaluated at certain points are given. Graphs are also given showing the variation of the stresses along the axes exterior to the rigid disk and on the boundary of the disk. Probably of more practical importance is the plate containing a rigid circular disk rather than a rigid elliptic disk. In the former case, the stresses for the corresponding problem involving an isotropic plate are plotted on the graphs for comparison.

CHAPTER II

THE EXACT SOLUTION FOR PLANE STRESS

The Mathematical Formulation of the Boundary Conditions

Choose the center of the ellipse as the origin of the coordinate system as shown in Figure 3. The axes of the ellipse are taken to coincide with the coordinate axes.

The boundary conditions may be stated mathematically using the notation of A. E. H. Love[1] as

$$X_x\big|_{x\to\pm\infty} = S, \qquad Y_y\big|_{y\to\pm\infty} = 0,$$

$$X_y\big|_{x\to\pm\infty} = 0, \qquad X_y\big|_{y\to\pm\infty} = 0,$$

and $u(x,y) = v(x,y) = 0$ on the circumference of the disk given by the parametric equations $x = a\cos\theta$, $y = b\sin\theta$. The displacements in the x and y directions are $u(x,y)$ and $v(x,y)$, respectively.

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[1] A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity (New York), 1944.

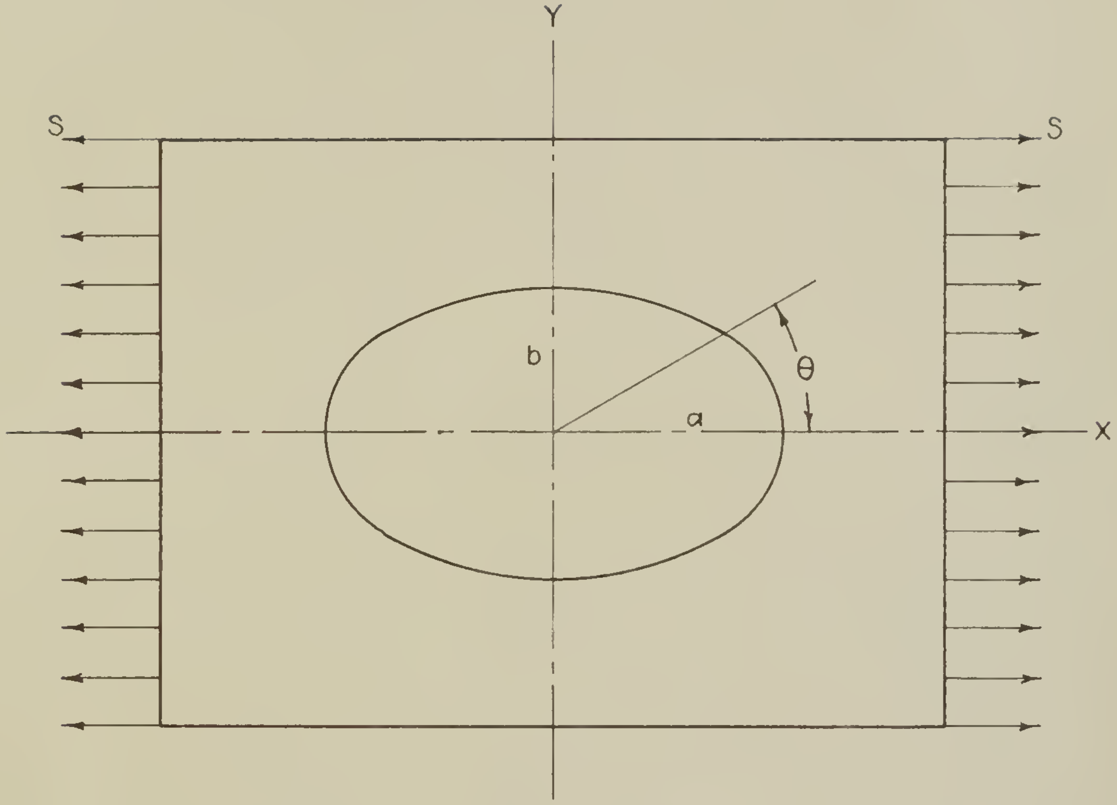

Figure 3.--Elliptic Disk in an Orthotropic Plate.

The Derivation of the Differential Equation for an Orthotropic Plate in a State of Plane Stress

If the axes of elastic symmetry are taken as the x, y, and z axes, the components of stress and strain are related by the following equations:[2]

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E_x} X_x - \frac{\sigma_{yx}}{E_y} Y_y - \frac{\sigma_{zx}}{E_z} Z_z \qquad (1)$$

$$e_{yy} = \frac{\partial v}{\partial y} = -\frac{\sigma_{xy}}{E_x} X_x + \frac{1}{E_y} Y_y - \frac{\sigma_{zy}}{E_z} Z_z \qquad (2)$$

$$e_{zz} = \frac{\partial w}{\partial z} = -\frac{\sigma_{xz}}{E_x} X_x - \frac{\sigma_{yz}}{E_y} Y_y + \frac{1}{E_z} Z_z \qquad (3)$$

$$e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{\mu_{xy}} X_y \qquad (4)$$

$$e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{1}{\mu_{yz}} Y_z \qquad (5)$$

$$e_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{1}{\mu_{zx}} Z_x \qquad (6)$$

In these equations E_x, E_y, and E_z are Young's moduli in the x, y, and z directions, respectively. The symbol σ represents Poisson's ratio; in particular, σ_{xy} is the ratio of the contraction parallel to the y-axis to the extension parallel to the y-axis associated with a tension parallel to the x-axis. The quantity μ is the modulus of rigidity; in particular, μ_{xy} is the modulus of rigidity associated with the directions of x and y. The displacements in the x, y, and z directions are u(x,y,z), v(x,y,z), and w(x,y,z), respectively.

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[2] H. W. March, Stress-Strain Relations in Wood and Plywood Considered as Orthotropic Materials, Forest Products Laboratory Report No. R1503, February, 1944, p. 2.

The problem here is a case of plane stress; that is, a thin plate is subjected to the action of forces applied to the boundary acting in the plane of the plate and distributed uniformly over the thickness of the plate. The stress components Z_z, Z_x, and Z_y are zero on both surfaces of the plate; and it can be assumed that these components are zero throughout the plate. The dimension z is then of no importance, and this dimension will be considered equal to unity.

For the assumptions of plane stress, equations (1), (2), and (4) reduce to:[3]

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E_x} X_x - \frac{\sigma_{yx}}{E_y} Y_y \tag{7}$$

$$e_{yy} = \frac{\partial v}{\partial y} = -\frac{\sigma_{xy}}{E_x} X_x + \frac{1}{E_y} Y_y \tag{8}$$

$$e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{\mu_{xy}} X_y . \tag{9}$$

---

[3] Ibid., p. 15.

The equations of equilibrium for plane stress are

$$\frac{\partial}{\partial x} X_x + \frac{\partial}{\partial y} X_y = 0, \qquad (10)$$

$$\frac{\partial}{\partial x} X_y + \frac{\partial}{\partial y} Y_y = 0. \qquad (11)$$

The equations of equilibrium are deduced from the conditions of static equilibrium with no reference to the laws relating stresses and strains. They will be satisfied by a stress function F(x,y)[4] such that

$$X_x = \frac{\partial^2 F}{\partial y^2}, \quad Y_y = \frac{\partial^2 F}{\partial x^2}, \quad X_y = -\frac{\partial^2 F}{\partial x \, \partial y}. \qquad (12)$$

The compatibility equation for plane stress,

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \, \partial y}, \qquad (13)$$

is derived using the concept that u(x,y) and v(x,y), the displacements in the x and y direction, are single-valued and continuous (class three).

Equations (12) substituted into equations (7), (8), and (9) give

$$e_{xx} = \frac{1}{E_x} \frac{\partial^2 F}{\partial y^2} - \frac{\sigma_{yx}}{E_y} \frac{\partial^2 F}{\partial x^2}, \qquad (14)$$

---

[4] G. B. Airy, Report of the British Association for the Advancement of Science, 1862.

$$e_{yy} = -\frac{\sigma_{xy}}{E_x}\frac{\partial^2 F}{\partial Y^2} + \frac{1}{E_y}\frac{\partial^2 F}{\partial X^2}, \tag{15}$$

$$e_{xy} = -\frac{1}{\mu_{xy}}\frac{\partial^2 F}{\partial X\,\partial Y}. \tag{16}$$

Combining equations (13), (14), (15), and (16) and using

$$\frac{\sigma_{yx}}{E_y} = \frac{\sigma_{xy}}{E_x}{}^{5} \tag{17}$$

leads to the differential equation,

$$\frac{1}{E_y}\frac{\partial^4 F}{\partial X^4} + \left(\frac{1}{\mu_{xy}} - \frac{2\sigma_{xy}}{E_x}\right)\frac{\partial^4 F}{\partial X^2 \partial Y^2} + \frac{1}{E_x}\frac{\partial^4 F}{\partial Y^4} = 0. \tag{18}$$

Equation (18) may be rewritten as

$$\frac{\partial^4 F}{\partial X^4} + 2K\frac{\partial^4 F}{\partial X^2 \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = 0 \tag{19}$$

where

$$K = \frac{\sqrt{E_x E_y}}{2}\left(\frac{1}{\mu_{xy}} - \frac{2\sigma_{xy}}{E_x}\right) \tag{20}$$

by the substitution

$$\eta = \epsilon Y,{}^{6} \tag{21}$$

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[5] March, op. cit., p. 7.

[6] H. W. March, Flat Plates of Plywood under Uniform or Concentrated Loads, Forest Products Laboratory Report No. 1312, March, 1942, p. 40.

where

$$\epsilon = \sqrt[4]{\frac{E_x}{E_y}}. \tag{22}$$

The General Solution of the Differential Equation

Substituting a solution of the form $F = F(x + \nu n)$ in equation (19) leads to the characteristic equation

$$\nu^4 + 2K\nu^2 + 1 = 0, \tag{23}$$

a quadratic in ν^2, or

$$\nu^2 = -K \pm \sqrt{K^2 - 1}. \tag{24}$$

For wood K as defined here is probably always greater than 1.[7]

Let

$$K = \cosh\phi \tag{25}$$

then

$$\begin{aligned} \nu^2 &= -\cosh\phi \pm \sqrt{\cosh^2\phi - 1} \\ &= -\cosh\phi \pm \sinh\phi \end{aligned}$$

---

[7] Smith, op. cit., p. 5.

$$= -\frac{e^{\phi}+e^{-\phi}}{2} \pm \frac{e^{\phi}-e^{-\phi}}{2}$$

$$= -e^{\pm\phi}.$$

The roots of equation (23) are

$$\nu_1 = i e^{\frac{\phi}{2}},\ \nu_2 = i e^{-\frac{\phi}{2}},\ \nu_3 = -i e^{\frac{\phi}{2}},\ \nu_4 = -i e^{-\frac{\phi}{2}}$$

where $i = \sqrt{-1}$. The differential equation (19) can now be written

$$\left(\frac{\partial^2}{\partial X^2} + \alpha^2\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial X^2} + \beta^2\frac{\partial^2}{\partial \eta^2}\right)F = 0 \tag{26}$$

where

$$\alpha^2 = e^{\phi},\quad \beta^2 = e^{-\phi} \tag{27}$$

since

$$\alpha^2 + \beta^2 = e^{\phi} + e^{-\phi} = 2\cosh\phi$$

$$= 2K. \tag{28}$$

It is important to note that

$$\nu_1 = i\alpha = i\sqrt{K+\sqrt{K^2-1}}, \tag{29}$$

$$\nu_2 = i\beta = i\sqrt{K-\sqrt{K^2-1}}, \tag{30}$$

$$\nu_3 = -i\alpha, \tag{31}$$

$$\nu_4 = -i\beta. \tag{32}$$

The general solution is

$$F = F_1(x+i\alpha\eta) + F_2(x-i\alpha\eta) + F_3(x+i\beta\eta) + F_4(x-i\beta\eta). \tag{33}$$

The stresses obtained from this stress function must be real. It will be shown that a real solution can be obtained from the complex solution. Consider

$$R\{F_1(x+i\alpha\eta)\}$$

which substituted in equation (19) gives

$$(1-2K\alpha^2+\alpha^4)\,R\{F_1^{iv}\}=0, \tag{34}$$

where the symbol

$$R\{\quad\}$$

denotes the real part of the quantity enclosed in the bracket. Equation (34) is satisfied by virtue of equations (23) and (29). Similar results are obtained with F_2, F_3, and F_4. Since

$$R\{f(z)\} = R\{f(\bar{z})\},$$

it follows that a real solution of the differential equation (19) is

$$R\{F\} = R\{F_{11}(x+i\alpha\eta) + F_{22}(x+i\beta\eta)\}, \tag{35}$$

where

$$F_{11} = F_1 + F_2 \quad \text{and} \quad F_{22} = F_3 + F_4 .$$

The Stresses Obtained from the Stress Function

Some work of Professor I. S. Sokolnikoff[8] on problems using isotropic material suggested the general form of the stress function. In the solution of the isotropic problems, a function of one complex variable is used. However, in the orthotropic case, a function of two complex variables is used. Several functions of two complex variables which satisfied the differential equation but did not satisfy the boundary conditions were tried. A suitable function given by Professor C. B. Smith[9], except here the constants are complex, is

$$F = R\left\{ \frac{A}{2\gamma_1^2}\left[\frac{1}{2}(z_1 - w_1)^2 + \gamma_1^2 \ln (z_1 + w_1)\right] + \frac{B}{2\gamma_2^2}\left[\frac{1}{2}(z_2 - w_2)^2 + \gamma_2^2 \ln (z_2 + w_2)\right]\right\} + \frac{S\eta^2}{2\epsilon^2} \qquad (36)$$

where

$$A = A_1 + i A_2, \quad B = B_1 + i B_2, \qquad (37)$$

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[8] I. S. Sokolnikoff, Mathematical Theory of Elasticity (Providence, Rhode Island), 1941, chap. v.

[9] Smith, op. cit., p. 6.

$$Z_1 = x + i\alpha\eta,\ Z_2 = x + i\beta\eta,\ \eta = \epsilon y, \tag{38}$$

$$W_1 = \sqrt{Z_1^2 - \gamma_1^2}, \qquad W_2 = \sqrt{Z_2^2 - \gamma_2^2}, \tag{39}$$

$$\gamma_1^2 = a^2 - \alpha^2\epsilon^2 b^2, \qquad \gamma_2^2 = a^2 - \beta^2\epsilon^2 b^2. \tag{40}$$

The boundary of the disk is given by

$$x = a\cos\theta,\ \eta = \epsilon y = \epsilon b \sin\theta. \tag{41}$$

S is the uniform tension applied at the ends of the plate.

In order that the stresses may be single-valued, the square roots in equation (39) are to be chosen so that the inequalities

$$|Z_1 + W_1| \geqq |\gamma_1| \quad \text{and} \quad |Z_2 + W_2| \geqq |\gamma_2| \tag{42}$$

are satisfied. These inequalities will be discussed further after the stresses are obtained.

The method used in deriving equation (19) establishes that any function satisfying equation (19) satisfies the equilibrium equations (10) and (11) and the compatibility equation (13). A solution for the problem must

(a) satisfy the differential equation (19);

(b) satisfy the exterior boundary conditions

$$X_x\big|_{x\to\pm\infty} = S, \qquad Y_y\big|_{y\to\pm\infty} = 0,$$

$$X_y\big|_{x\to\pm\infty} = 0, \qquad X_y\big|_{y\to\pm\infty} = 0;$$

(c) satisfy the interior boundary conditions; that is, the displacements u and v are zero along the boundary of the disk

$$X = a\cos\theta, \quad \eta = \epsilon Y = \epsilon b \sin\theta.$$

The following formulas will be found useful throughout this dissertation.

$$\frac{\partial f(Z_1)}{\partial X} = \frac{\partial f}{\partial Z_1}\frac{\partial Z_1}{\partial X} = \frac{\partial f}{\partial Z_1}$$

$$\frac{\partial f(Z_2)}{\partial X} = \frac{\partial f}{\partial Z_2}\frac{\partial Z_2}{\partial X} = \frac{\partial f}{\partial Z_2}$$

$$\frac{\partial f(Z_1)}{\partial \eta} = \frac{\partial f}{\partial Z_1}\frac{\partial Z_1}{\partial \eta} = i\alpha\frac{\partial f}{\partial Z_1} \qquad (43)$$

$$\frac{\partial f(Z_2)}{\partial \eta} = \frac{\partial f}{\partial Z_2}\frac{\partial Z_2}{\partial \eta} = i\beta\frac{\partial f}{\partial Z_2}$$

$$\frac{\partial W_1}{\partial Z_1} = \frac{Z_1}{W_1} \qquad \frac{\partial W_2}{\partial Z_2} = \frac{Z_2}{W_2}$$

It follows that

$$\frac{\partial F}{\partial X} = R\left\{\frac{A}{2\gamma_1^2}\left[(Z_1-W_1)\left(1-\frac{Z_1}{W_1}\right)+\gamma_1^2\frac{1+\frac{Z_1}{W_1}}{Z_1+W_1}\right]\right.$$

$$\left.+\frac{B}{2\gamma_2^2}\left[(Z_2-W_2)\left(1-\frac{Z_2}{W_2}\right)+\gamma_2^2\frac{1+\frac{Z_2}{W_2}}{Z_2+W_2}\right]\right\} \qquad (44)$$

$$= R\left\{ \frac{A}{2\gamma_1^2}\left[\frac{-(Z_1-W_1)^2}{W_1} + \frac{\gamma_1^2}{W_1}\right] + \frac{B}{2\gamma_2^2}\left[\frac{-(Z_2-W_2)^2}{W_2} + \frac{\gamma_2^2}{W_2}\right]\right\}. \tag{45}$$

From equations (45) and (39)

$$\frac{\partial F}{\partial X} = R\left\{ \frac{A}{2}\left[\frac{-(Z_1-W_1)}{W_1(Z_1+W_1)} + \frac{Z_1+W_1}{W_1(Z_1+W_1)}\right] + \frac{B}{2}\left[\frac{-(Z_2-W_2)}{W_2(Z_2+W_2)} + \frac{Z_2+W_2}{W_2(Z_2+W_2)}\right]\right\}$$

$$= R\left\{ \frac{A}{Z_1+W_1} + \frac{B}{Z_2+W_2}\right\}. \tag{46}$$

Equations (46) and (12) lead to

$$Y_y = \frac{\partial^2 F}{\partial X^2} = R\left\{ \frac{-A}{W_1(Z_1+W_1)} + \frac{-B}{W_2(Z_2+W_2}\right\} \tag{47}$$

By equations (12), (21), and (43)

$$X_x = \epsilon^2 \frac{\partial^2 F}{\partial \eta^2} = R\left\{ \frac{\alpha^2\epsilon^2 A}{W_1(Z_1+W_1)} + \frac{\beta^2\epsilon^2 B}{W_2(Z_2+W_2)}\right\} + S \tag{48}$$

and

$$X_y = -\epsilon\frac{\partial^2 F}{\partial x\,\partial \eta} = R\left\{ \frac{i\alpha\epsilon A}{W_1(Z_1+W_1)} + \frac{i\beta\epsilon B}{W_2(Z_2+W_2)}\right\} \tag{49}$$

may be written immediately.

The stress equations (47), (48), and (49) contain multiple valued expressions of the form

$$\frac{1}{W_1(Z_1+W_1)} \quad \text{and} \quad \frac{1}{W_2(Z_2+W_2)}.$$

γ_1^2 and γ_2^2 defined by equation (40) may be positive or negative, depending upon the orthotropic material, orientation of the axes, and the values of a and b. Assume γ_1^2 is less than zero, say γ_1^2 equals $-c^2$. Define

$$W = Z_1 + W_1 = Z_1 \pm \sqrt{Z_1^2 + c^2}. \qquad (50)$$

The branch points of the function W are ±ic. A straight line connecting the points +ic and -ic may be taken as the branch-cut. The branch-cut in the Z_1-plane can be mapped onto the W-plane in the following manner.

Define

$$X_1 = X = 0 \quad \text{and} \quad Y_1 = \kappa \epsilon Y.$$

Then

$$W = U + iV = iY_1 \pm \sqrt{c^2 - Y_1^2}$$

where

$$|Y_1| \leq |c|.$$

Hence

$$U = \pm\sqrt{c^2 - Y_1^2}, \quad V = Y_1$$

and $u^2 + v^2 = c^2$ is a circle with center at the origin of the W-plane and radius $|c|$.

The inequality

$$|W| = |Z_1 + W_1| \geqq |c| = |\gamma_1|,$$

relation (42) given before, restricts the variable Z_1 from following a path crossing the branch-cut and determines which value to assign W. Hence, by means of the function W, subject to the relation (42), the entire Z_1-plane with the exception of the branch-cut maps into the region of the W-plane exterior to the circle

$$W = |\gamma_1| e^{i\theta}.$$

If γ_1^2 is greater than zero, it can be shown that the function W has the branch points $\pm\gamma_1$. A straight line connecting the points $+\gamma_1$ and $-\gamma_1$ in the Z_1-plane may be taken as the branch-cut. Here again, by means of the function W, subject to the relation (42), the entire Z_1-plane with the exception of the branch-cut maps into the region of the W-plane exterior to the circle

$$W = |\gamma_1| e^{i\theta}.$$

Similarly for the expression with the subscript 2.

The stresses at any point of the plate except those interior points of the elliptic disk are given by the equations (47), (48), and (49). The values of the radicals defined in equations (39) are determined for the point by the inequalities (42).

The Exterior Boundary Conditions Satisfied

It will be shown that the exterior boundary conditions are satisfied for any finite value of the complex constants A and B. These constants are uniquely determined by the condition that the displacements are zero on the boundary of the rigid elliptic disk.

In the previous section it was shown that the function $W = Z_1 + W_1$, subject to the relation (42), maps the entire Z_1-plane with the exception of the branch-cut into the W-plane exterior to the circle

$$W = |\gamma_1| e^{i\theta} .$$

It also follows that the point $Z_1 = \infty$ maps into the point $W = Z_1 + W_1 = \infty$. $Z_1 = \infty$ for the edges of the plate $x = \pm\infty$. Similar results are obtained with the terms of subscript 2.

It follows from the above remarks and equations (48) and (49) that on the two edges $x = \pm\infty$

$$X_x \Big|_{\substack{Z_1 \to \infty \\ Z_2 \to \infty}} = R\left\{ 0 + 0 \right\} + S = S$$

and

$$X_y \Big|_{\substack{Z_1 \to \infty \\ Z_2 \to \infty}} = R\left\{ 0 + 0 \right\} = 0.$$

These last two expressions are the exterior boundary conditions for the edges $x = \pm\infty$.

Also $Z_1 = \infty$ for the edges of the plate $y = \pm \infty$. Again $W = Z_1 + W_1 = \infty$ evaluated along the edges of the plate $y = \pm\infty$. Similar results are obtained with the terms of subscript 2.

It follows from these remarks and equations (47) and (49) that along the edge of the plate perpendicular to the y-axis

$$\underline{Y}_y \Big|_{\substack{Z_1 \to \infty \\ Z_2 \to \infty}} = R\Big\{ 0 + 0 \Big\} = 0$$

and

$$\underline{X}_y \Big|_{\substack{Z_1 \to \infty \\ Z_2 \to \infty}} = R\Big\{ 0 + 0 \Big\} = 0.$$

These last two expressions are the exterior boundary conditions for the edges $y = \pm\infty$.

The Derivation of the Displacements u and v

Substitution of the stresses given by equations (47) and (48) into equation (7) leads to

$$\frac{\partial u}{\partial x} = e_{xx} = R\Big\{A\Big[\frac{\kappa^2\epsilon^2}{E_x} + \frac{\sigma_{yx}}{E_y}\Big]\frac{1}{W_1(Z_1+W_1)} + B\Big[\frac{\beta^2\epsilon^2}{E_x} + \frac{\sigma_{yx}}{E_y}\Big]\frac{1}{W_2(Z_2+W_2)}\Big\} + \frac{S}{E_x}. \qquad (51)$$

Integrating,

$$U = R\left\{A\left[\frac{\alpha^2\epsilon^2}{E_x} + \frac{\sigma_{yx}}{E_y}\right]\left[\frac{-1}{Z_1+W_1}\right] + B\left[\frac{\beta^2\epsilon^2}{E_x} + \frac{\sigma_{yx}}{E_y}\right]\right.$$

$$\left.\left[\frac{-1}{Z_2+W_2}\right]\right\} + \frac{SX}{E_x} + U_o(Y) \qquad (52)$$

where $u_o(y)$ is an arbitrary function and will be shown to be identically zero.

Substitution of the stresses given by equations (47) and (48) into equation (8) leads to

$$\frac{\partial v}{\partial y} = e_{yy} = R\left\{A\left[\frac{1}{E_y} + \frac{\alpha^2\epsilon^2\sigma_{xy}}{E_x}\right]\frac{-1}{W_1(Z_1+W_1)}\right.$$

$$\left. + B\left[\frac{1}{E_y} + \frac{\beta^2\epsilon^2\sigma_{xy}}{E_x}\right]\frac{-1}{W_2(Z_2+W_2)}\right\}$$

$$- \frac{S\sigma_{xy}}{E_x}. \qquad (53)$$

Integrating,

$$V = R\left\{A\left[\frac{1}{E_y} + \frac{\alpha^2\epsilon^2\sigma_{xy}}{E_x}\right]\left[\frac{1}{i\alpha\epsilon(Z_1+W_1)}\right] + B\left[\frac{1}{E_y} + \frac{\beta^2\epsilon^2\sigma_{xy}}{E_x}\right]\right.$$

$$\left.\left[\frac{1}{i\beta\epsilon(Z_2+W_2)}\right]\right\} - \frac{SY\sigma_{xy}}{E_x} + V_o(X) \qquad (54)$$

where $v_o(x)$ is an arbitrary function and will be shown to be identically zero.

It follows from equations (52) and (17) that

$$U = R\left\{-\frac{g_{11}A}{Z_1+W_1} - \frac{g_{12}B}{Z_2+W_2} + \frac{Sx}{E_x}\right\} + U_0(Y) \tag{55}$$

where

$$g_{11} = \frac{\epsilon^2\alpha^2 + \sigma_{xy}}{E_x} \quad \text{and} \quad g_{12} = \frac{\epsilon^2\beta^2 + \sigma_{xy}}{E_x}. \tag{56}$$

From equations (53) and (17), we have

$$V = R\left\{\frac{g_{21}A}{i\alpha\epsilon(Z_1+W_1)} + \frac{g_{22}B}{i\beta\epsilon(Z_2+W_2)} - \frac{SY\sigma_{yx}}{E_y}\right\}$$

$$+ V_0(X) \tag{57}$$

where

$$g_{21} = \frac{1+\sigma_{yx}\epsilon^2\alpha^2}{E_y} \quad \text{and} \quad g_{22} = \frac{1+\sigma_{yx}\beta^2\epsilon^2}{E_y}. \tag{58}$$

Substitution of the derivatives obtained from equations (55) and (57) as required by equation (9) leads to

$$X_y = R\left\{\mu_{xy}\left[\frac{g_{11}Ai\alpha\epsilon}{W_1(Z_1+W_1)} + \frac{g_{12}Bi\beta\epsilon}{W_2(Z_2+W_2)}\right.\right.$$

$$\left.\left. - \frac{g_{21}A}{i\alpha\epsilon W_1(Z_1+W_1)} - \frac{g_{22}B}{i\beta\epsilon W_2(Z_2+W_2)}\right]\right\}$$

$$+\mu_{xy}\left(\frac{d}{dY}U_0(Y) + \frac{d}{dX}V_0(X)\right) \tag{59}$$

$$= R\left\{ \mu_{xy}\left(i\alpha\epsilon g_{11} - \frac{g_{21}}{i\alpha\epsilon}\right)\frac{A}{W_1(Z_1+W_1)} + \mu_{xy}\left(i\beta\epsilon g_{12} - \frac{g_{22}}{i\beta\epsilon}\right)\frac{B}{W_2(Z_2+W_2)}\right\} + \mu_{xy}\left(\frac{d}{dy}U_0 + \frac{d}{dx}V_0\right). \tag{60}$$

Now

$$\mu_{xy}\left(i\alpha\epsilon g_{11} - \frac{g_{21}}{i\alpha\epsilon}\right) = \mu_{xy}\left(\frac{i\alpha^3\epsilon^3 + i\alpha\epsilon\sigma_{xy}}{E_x} - \frac{1+\sigma_{yx}\epsilon^2\alpha^2}{i\alpha\epsilon E_y}\right). \tag{61}$$

It follows from equation (22) that

$$E_x = \epsilon^4 E_y,$$

hence

$$\mu_{xy}\left(i\alpha\epsilon g_{11} - \frac{g_{21}}{i\alpha\epsilon}\right) = \frac{\mu_{xy}}{E_y}\left(\frac{i\alpha^3\epsilon^3 + i\alpha\epsilon\sigma_{xy}}{\epsilon^4} - \frac{1+\sigma_{yx}\epsilon^2\alpha^2}{i\alpha\epsilon}\right). \tag{62}$$

The relation

$$\sigma_{yx} = \frac{E_y}{E_x}\sigma_{xy} = \frac{\sigma_{xy}}{\epsilon^4}$$

is obtained from equations (17) and (22).

Hence,

$$\mu_{xy}\left(i\alpha\epsilon g_{11} - \frac{g_{21}}{i\alpha\epsilon}\right) = \frac{\mu_{xy}}{E_y}\left(\frac{i\alpha^3\epsilon^3 + i\alpha\epsilon\sigma_{xy}}{\epsilon^4} - \frac{1 + \frac{\sigma_{xy}}{\epsilon^4}\alpha^2\epsilon^2}{i\alpha\epsilon}\right)$$

$$= \frac{\mu_{xy}}{\epsilon^4 E_y i\alpha\epsilon}\left(-\alpha^4\epsilon^4 - \epsilon^4 - 2\sigma_{xy}\alpha^2\epsilon^2\right). \qquad (63)$$

From equation (27), we have

$$\alpha^2\beta^2 = 1.$$

Then

$$\mu_{xy}\left(i\alpha\epsilon g_{11} - \frac{g_{21}}{i\alpha\epsilon}\right) = \frac{\epsilon^2\mu_{xy}}{\epsilon^4 E_y i\alpha\epsilon}\left(-\frac{\alpha^2\epsilon^2}{\beta^2} - \epsilon^2 - \frac{2\sigma_{xy}}{\beta^2}\right)$$

$$= \frac{-\mu_{xy}}{\beta^2\epsilon^2 E_y i\alpha\epsilon}\left(\alpha^2\epsilon^2 + \beta^2\epsilon^2 + 2\sigma_{xy}\right). \qquad (64)$$

Equations (20) and (22) combine to give

$$2K = \frac{\sqrt{E_x E_y}}{\mu_{xy}} - \frac{2\sigma_{xy}}{\epsilon^2} \qquad (65)$$

or

$$\frac{\sqrt{E_x E_y}}{\mu_{xy}} = 2K + \frac{2\sigma_{xy}}{\epsilon^2}; \qquad (66)$$

and substituting from equation (28), equation (66) is

$$\frac{\sqrt{E_x E_y}}{\mu_{xy}} = \alpha^2 + \beta^2 + \frac{2\sigma_{xy}}{\epsilon^2}$$

$$= \frac{\alpha^2\epsilon^2 + \beta^2\epsilon^2 + 2\sigma_{xy}}{\epsilon^2}. \qquad (67)$$

Equation (64) by use of the results of equation (67) reduces to

$$\mu_{xy}\left(i\alpha\epsilon g_{11} - \frac{g_{21}}{i\alpha\epsilon}\right) = \frac{i\alpha}{\epsilon}\sqrt{\frac{E_x}{E_y}} = i\alpha\epsilon \qquad (68)$$

In an analogous manner, it follows that

$$\mu_{xy}\left(i\beta\epsilon g_{12} - \frac{g_{22}}{i\beta\epsilon}\right) = i\beta\epsilon. \qquad (69)$$

Equations (68) and (69) substituted into equations (60) reduces it to

$$X_y = R\left\{\frac{i\alpha\epsilon A}{W_1(Z_1+W_1)} + \frac{i\beta\epsilon B}{W_2(Z_2+W_2)}\right\}$$

$$+ \mu_{xy}\left(\frac{du_0}{dy} + \frac{dv_0}{dx}\right). \qquad (70)$$

It is seen by direct comparison of equations (49) and (70) that the above value for the shear and the shear obtained from the stress function are equal if and only if

$$\mu_{xy}\left(\frac{du_0}{dy} + \frac{dv_0}{dx}\right) = 0; \qquad (71)$$

but

$$\mu_{xy} \neq 0$$

hence

$$\frac{du_o}{dy} + \frac{dv_o}{dx} = 0. \qquad (72)$$

Now du_o/dy is a function of y alone, and dv_o/dx is a function of x alone. It follows that

$$\frac{du_o}{dy} = k \qquad \text{and} \qquad \frac{dv_o}{dx} = -k$$

where k is an arbitrary constant.

Integrating,

$$u_o = ky + m \qquad \text{and} \qquad v_o = -kx + n$$

where m and n are arbitrary constants. But m = n = 0 since the displacements are zero along the boundary of the rigid elliptic disk for all values of θ. Further, the plate is symmetrical about the x-axis, and the stresses are applied symmetrically; therefore, the displacements must be symmetrical with respect to the x-axis. This is expressed mathematically by requiring $u_o(y)$ to be an even function. It follows then that k must be zero. The arbitrary function $u_o(y)$ is identically zero.

By a similar argument, it can be shown that the arbitrary function $v_o(x)$ is also identically zero.

A and B Determined from the Interior Boundary Conditions

The following quantities are evaluated on the boundary of the rigid elliptic disk.[10]

$$W_1 = \kappa \epsilon b \cos\theta + i a \sin\theta \tag{73}$$

$$W_2 = \beta \epsilon b \cos\theta + i a \sin\theta \tag{74}$$

$$Z_1 + W_1 = (a + \kappa \epsilon b)\, e^{i\theta} \tag{75}$$

$$Z_2 + W_2 = (a + \beta \epsilon b)\, e^{i\theta} \tag{76}$$

Substitution of the equations (75) and (76) into equation (52), recalling that $u_o(y)$ is identically zero and that the displacements on the boundary of the rigid disk are zero, gives

$$\begin{aligned} U = R\Bigg\{ & A\left[\frac{\epsilon^2\kappa^2}{E_x} + \frac{\sigma_{yx}}{E_y}\right]\frac{-e^{-i\theta}}{a+\kappa\epsilon b} \\ & + B\left[\frac{\epsilon^2\beta^2}{E_x} + \frac{\sigma_{yx}}{E_y}\right]\frac{-e^{-i\theta}}{a+\beta\epsilon b}\Bigg\} \\ & + \frac{S a \cos\theta}{E_x} = 0 \end{aligned} \tag{77}$$

$$= \left(\frac{-A_1}{a+\kappa\epsilon b}\left[\frac{\epsilon^2\kappa^2}{E_x} + \frac{\sigma_{yx}}{E_y}\right] + \frac{-B_1}{a+\beta\epsilon b}\left[\frac{\epsilon^2\beta^2}{E_x} + \frac{\sigma_{yx}}{E_y}\right] \right.$$

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[10] Smith, *op. cit.*, p. 7.

$$+\frac{Sa}{E_x}\Big)\cos\theta+\Big(\frac{-A_2}{a+\alpha\epsilon b}\left[\frac{\epsilon^2\alpha^2}{E_x}+\frac{\sigma_{yx}}{E_y}\right]$$

$$+\frac{-B_2}{a+\beta\epsilon b}\left[\frac{\epsilon^2\beta^2}{E_x}+\frac{\sigma_{yx}}{E_y}\right]\Big)\sin\theta=0. \qquad (78)$$

Since $u \equiv 0$ for $0 \leqq \theta \leqq 2\pi$, it follows that

$$\frac{A_1}{a+\alpha\epsilon b}\left[\frac{\epsilon^2\alpha^2}{E_x}+\frac{\sigma_{yx}}{E_y}\right]+\frac{B_1}{a+\beta\epsilon b}\left[\frac{\epsilon^2\beta^2}{E_x}+\frac{\sigma_{yx}}{E_y}\right]$$

$$-\frac{Sa}{E_x}=0 \qquad (79)$$

and

$$\frac{A_2}{a+\alpha\epsilon b}\left[\frac{\epsilon^2\alpha^2}{E_x}+\frac{\sigma_{yx}}{E_y}\right]+\frac{B_2}{a+\beta\epsilon b}\left[\frac{\epsilon^2\beta^2}{E_x}+\frac{\sigma_{yx}}{E_y}\right]=0. \qquad (80)$$

Rewriting equation (79) with the use of equation (17), it becomes

$$\frac{\epsilon^2\alpha^2+\sigma_{xy}}{a+\alpha\epsilon b}A_1+\frac{\epsilon^2\beta^2+\sigma_{xy}}{a+\beta\epsilon b}B_1-Sa=0. \qquad (81)$$

Substitution of the equations (75) and (76) into equation (54), recalling that $v_0(x)$ is identically zero and that the displacements on the boundary of the rigid disk are zero, gives

$$V=R\Big\{A\left[\frac{1}{E_y}+\frac{\alpha^2\epsilon^2\sigma_{xy}}{E_x}\right]\frac{-ie^{-i\theta}}{\alpha\epsilon(a+\alpha\epsilon b)}$$

$$+B\left[\frac{1}{E_y}+\frac{\beta^2\epsilon^2\sigma_{xy}}{E_x}\right]\frac{-i e^{-i\theta}}{\beta\epsilon(a+\beta\epsilon b)}\Bigg\}$$

$$-\frac{\sigma_{xy}}{E_x}Sb\sin\theta=0 \qquad (82)$$

$$=\left(\left[\frac{-A_1}{\alpha\epsilon(a+\alpha\epsilon b)}\right]\left[\frac{1}{E_y}+\frac{\alpha^2\epsilon^2\sigma_{xy}}{E_x}\right]+\left[\frac{-B_1}{\beta\epsilon(a+\beta\epsilon b)}\right]\right.$$

$$\left.\left[\frac{1}{E_y}+\frac{\beta^2\epsilon^2\sigma_{xy}}{E_x}\right]-\frac{\sigma_{xy}}{E_x}Sb\right)\sin\theta$$

$$+\left(\left[\frac{-A_2}{\alpha\epsilon(a+\alpha\epsilon b)}\right]\left[\frac{1}{E_y}+\frac{\alpha^2\epsilon^2\sigma_{xy}}{E_x}\right]+\left[\frac{-B_2}{\beta\epsilon(a+\beta\epsilon b)}\right]\right.$$

$$\left.\left[\frac{1}{E_y}+\frac{\beta^2\epsilon^2\sigma_{xy}}{E_x}\right]\right)\cos\theta=0. \qquad (83)$$

Since $v \equiv 0$ for $0 \leqq \theta \leqq 2\pi$, it follows that

$$A_1\left[\frac{1}{E_y}+\frac{\alpha^2\epsilon^2\sigma_{xy}}{E_x}\right]\left[\frac{1}{\alpha\epsilon(a+\alpha\epsilon b)}\right]+B_1\left[\frac{1}{E_y}+\frac{\beta^2\epsilon^2\sigma_{xy}}{E_x}\right]$$

$$\left[\frac{1}{\beta\epsilon(a+\beta\epsilon b)}\right]+\frac{\sigma_{xy}}{E_x}Sb=0 \qquad (84)$$

and

$$A_2\left[\frac{1}{E_y}+\frac{\alpha^2\epsilon^2\sigma_{xy}}{E_x}\right]\left[\frac{1}{\alpha\epsilon(a+\alpha\epsilon b)}\right]+B_2\left[\frac{1}{E_y}+\frac{\beta^2\epsilon^2\sigma_{xy}}{E_x}\right]$$

$$\left[\frac{1}{\beta\epsilon(a+\beta\epsilon b)}\right]=0 \qquad (85)$$

With the use of equation (17), equation (84) may be rewritten as

$$\frac{1+\alpha^2\epsilon^2\sigma_{\overline{yx}}}{\alpha\epsilon(a+\alpha\epsilon b)}A_1+\frac{1+\beta^2\epsilon^2\sigma_{\overline{yx}}}{\beta\epsilon(a+\beta\epsilon b)}B_1+\sigma_{\overline{yx}}Sb=0 \tag{86}$$

Solving equations (80) and (85) simultaneously,

$$A_2=B_2=0. \tag{87}$$

Solving equations (81) and (86) simultaneously,

$$A_1=\frac{\sigma_{\overline{yx}}Sb(\epsilon^2\beta^2+\sigma_{\overline{xy}})+\dfrac{Sa}{\beta\epsilon}(1+\epsilon^2\beta^2\sigma_{\overline{yx}})}{\dfrac{(1+\epsilon^2\beta^2\sigma_{\overline{yx}})(\epsilon^2\alpha^2+\sigma_{\overline{xy}})}{\beta\epsilon(a+\alpha\epsilon b)}-\dfrac{(1+\alpha^2\epsilon^2\sigma_{\overline{yx}})(\epsilon^2\beta^2+\sigma_{\overline{xy}})}{\alpha\epsilon(a+\alpha\epsilon b)}}$$

or, with the use of $\alpha\beta = 1$ obtained from equations (27),

$$A_1=\frac{\dfrac{Sa\alpha}{\epsilon}(1+\epsilon^2\beta^2\sigma_{\overline{yx}})+Sb\sigma_{\overline{yx}}(\epsilon^2\beta^2+\sigma_{\overline{xy}})}{\dfrac{\alpha(\epsilon^2\alpha^2+\sigma_{\overline{xy}})(1+\epsilon^2\beta^2\sigma_{\overline{yx}})}{\epsilon(a+\alpha\epsilon b)}-\dfrac{\beta(1+\epsilon^2\alpha^2\sigma_{\overline{yx}})(\epsilon^2\beta^2+\sigma_{\overline{xy}})}{\epsilon(a+\alpha\epsilon b)}}; \tag{88}$$

and,

$$B_1=\frac{\sigma_{\overline{yx}}Sb(\epsilon^2\alpha^2+\sigma_{\overline{xy}})+\dfrac{Sa}{\alpha\epsilon}(1+\epsilon^2\alpha^2\sigma_{\overline{yx}})}{\dfrac{(1+\epsilon^2\alpha^2\sigma_{\overline{yx}})(\epsilon^2\beta^2+\sigma_{\overline{xy}})}{\alpha\epsilon(a+\beta\epsilon b)}-\dfrac{(1+\epsilon^2\beta^2\sigma_{\overline{yx}})(\epsilon^2\alpha^2+\sigma_{\overline{xy}})}{\beta\epsilon(a+\beta\epsilon b)}}$$

or

$$B_1=-\frac{\dfrac{Sa\beta}{\epsilon}(1+\epsilon^2\alpha^2\sigma_{\overline{yx}})+Sb\sigma_{\overline{yx}}(\epsilon^2\alpha^2+\sigma_{\overline{xy}})}{\dfrac{\alpha(\epsilon^2\alpha^2+\sigma_{\overline{xy}})(1+\epsilon^2\beta^2\sigma_{\overline{yx}})}{\epsilon(a+\beta\epsilon b)}-\dfrac{\beta(1+\epsilon^2\alpha^2\sigma_{\overline{yx}})(\epsilon^2\beta^2+\sigma_{\overline{xy}})}{\epsilon(a+\beta\epsilon b)}}. \tag{89}$$

The equations (87), (88), (89), and (36) lead to the stress function

$$\begin{aligned}
F = R\Bigg\{ & \frac{1}{2\gamma_1^2}\left[\frac{\dfrac{Sa\alpha}{\epsilon}(1+\epsilon^2\beta^2\sigma_{yx}) + Sb\sigma_{yx}(\epsilon^2\beta^2+\sigma_{xy})}{\dfrac{\alpha(\epsilon^2\alpha^2+\sigma_{xy})(1+\epsilon^2\beta^2\sigma_{yx})}{\epsilon(a+\alpha\epsilon b)} - \dfrac{\beta(1+\epsilon^2\alpha^2\sigma_{yx})(\epsilon^2\beta^2+\sigma_{xy})}{\epsilon(a+\alpha\epsilon b)}}\right] \cdot \\
& \left[\frac{1}{2}(Z_1-W_1)^2+\gamma_1^2 \ln(Z_1+W_1)\right] \\
& - \frac{1}{2\gamma_2^2}\left[\frac{\dfrac{Sa\beta}{\epsilon}(1+\epsilon^2\alpha^2\sigma_{yx}) + Sb\sigma_{yx}(\epsilon^2\alpha^2+\sigma_{xy})}{\dfrac{\alpha(\epsilon^2\alpha^2+\sigma_{xy})(1+\epsilon^2\beta^2\sigma_{yx})}{\epsilon(a+\beta\epsilon b)} - \dfrac{\beta(1+\epsilon^2\alpha^2\sigma_{yx})(\epsilon^2\beta^2+\sigma_{xy})}{\epsilon(a+\beta\epsilon b)}}\right] \\
& \left[\frac{1}{2}(Z_2-W_2)^2+\gamma_2^2 \ln(Z_2+W_2)\right]\Bigg\} + \frac{Sh^2}{2\epsilon^2}.
\end{aligned} \tag{[illegible]}$$

The Isotropic Stress Function as the Limiting Case of the Orthotropic Stress Function

It is desirable to show that the orthotropic stress function contains as a special case the isotropic stress function for a = b. First, set $\epsilon = 1$. By virtue of equation (22), $E_x = E_y$; but, this is not necessarily so for every orientation of the axes. $E_x = E_y$ is a necessary condition for the isotropic solution, but it is not a sufficient condition.

It follows from equation (17) that we can write

$$\sigma_{yx} = \sigma_{xy} = \sigma.$$

By virtue of the above remarks and equation (40), the stress function (90) simplifies to the expression

$$F=R\Big\{\Big[\frac{Sa\alpha(1+\beta^2\sigma)+Sb\sigma(\beta^2+\sigma)}{\alpha(\alpha^2+\sigma)(1+\beta^2\sigma)-\beta(1+\alpha^2\sigma)(\beta^2+\sigma)}\Big]\Big[\frac{1}{2(a-\alpha b)}\Big]$$

$$\Big[\frac{1}{2}(Z_1-W_1)^2+\gamma_1^2\ln(Z_1+W_1)\Big]$$

$$-\Big[\frac{Sa\beta(1+\alpha^2\sigma)+Sb\sigma(\alpha^2+\sigma)}{\alpha(\alpha^2+\sigma)(1+\beta^2\sigma)-\beta(1+\alpha^2\sigma)(\beta^2+\sigma)}\Big]\Big[\frac{1}{2(a-\beta b)}\Big]$$

$$\Big[\frac{1}{2}(Z_2-W_2)^2+\gamma_2^2\ln(Z_2+W_2)\Big]\Big\}+\frac{Sy^2}{2} \qquad (91)$$

where $Z_1 = x + i\alpha y$ and $Z_2 = x + i\beta y$.

Rewriting,

$$F=R\Big\{\Big[\frac{1}{\alpha(\alpha^2+\sigma)(1+\beta^2\sigma)-\beta(1+\alpha^2\sigma)(\beta^2+\sigma)}\Big]\Big[\Big(\frac{1}{2(a-\alpha b)}\Big)$$

$$\Big(Sa\alpha(1+\beta^2\sigma)+Sb\sigma(\beta^2+\sigma)\Big)\Big(\frac{1}{2}(Z_1-W_1)^2+\gamma_1^2\ln(Z_1+W_1)\Big)$$

$$-\Big(\frac{1}{2(a-\beta b)}\Big)\Big(Sa\beta(1+\alpha^2\sigma)+Sb\sigma(\alpha^2+\sigma)\Big)$$

$$\Big(\frac{1}{2}(Z_2-W_2)^2+\gamma_2^2\ln(Z_2+W_2)\Big)\Big]\Big]\Big\}+\frac{Sy^2}{2}. \qquad (92)$$

For the isotropic case, it is now sufficient that $\phi \longrightarrow 0$; since, by equation (23) $\alpha \longrightarrow 1$ and equation (19) reduces to the biharmonic equation for the isotropic case.

The following limits will be found useful.

$$\lim_{\phi\to 0} Z_1 = \lim_{\phi\to 0} Z_2 = X + iY = Z$$

$$\lim_{\phi\to 0} \gamma_1^2 = \lim_{\phi\to 0} \gamma_2^2 = a^2 - b^2 = \gamma^2$$

$$\lim_{\phi\to 0} W_1 = \lim_{\phi\to 0} W_2 = \sqrt{Z^2 - \gamma^2} = W$$

$$\lim_{\phi\to 0} \alpha = \lim_{\phi\to 0} \beta = 1$$

If $\phi = 0$, the bracketed expression of equation (92) is indeterminate. In order to evaluate the indeterminate expression, first rewrite equation (92) with the use of equations (27). This leads to

$$F = R\left\{\left[\frac{1}{e^{\phi/2}(e^{\phi}+\sigma)(1+e^{-\phi}\sigma) - e^{-\phi/2}(1+e^{\phi}\sigma)(e^{-\phi}+\sigma)}\right]\right.$$

$$\left[\left(\frac{1}{2(a-e^{\phi/2}b)}\right)\left(Sae^{\phi/2}(1+e^{-\phi}\sigma)+Sb\sigma(e^{-\phi}+\sigma)\right)\right.$$

$$\left(\frac{1}{2}(Z_1-W_1)^2+\gamma_1^2\ln(Z_1+W_1)\right) - \left(\frac{1}{2(a-e^{-\phi/2}b)}\right)$$

$$\left(Sa\,e^{-\phi/2}(1+e^{\phi}\sigma)+Sb\sigma(e^{\phi}+\sigma)\right)$$

$$\left(\tfrac{1}{2}(Z_2-W_2)^2+\gamma_2^2\ln(Z_2+W_2)\right)\Big]\Big]\Big\}+\frac{Sy^2}{2} \qquad (93)$$

where

$$\gamma_1^2=a^2-e^{\phi}b^2,\quad \gamma_2^2=a^2-e^{-\phi}b^2,$$

$$Z_1=x+ie^{\phi/2}y,\quad Z_2=x+ie^{-\phi/2}\boxed{b^2.}\qquad \gamma$$

Evaluating by differentiating both numerator and denominator with respect to ϕ, it follows that

$$F=R\left\{\left[\frac{1}{U-V}\right]\left[\left(-\frac{-\frac{1}{2}e^{\phi/2}b}{2(a-e^{\phi/2}b)^2}\right)\left(Sa\,e^{\phi/2}(1+e^{-\phi}\sigma)\right.\right.\right.$$

$$\left.+Sb\sigma(e^{-\phi}+\sigma)\right)\left(\tfrac{1}{2}(Z_1-W_1)^2+\gamma_1^2\ln(Z_1+W_1)\right)$$

$$+\left(\frac{1}{2(a-e^{\phi/2}b)}\right)\left(\frac{Sa}{2}e^{\phi/2}(1+e^{-\phi}\sigma)+Sa\,e^{\phi/2}\right.$$

$$(-e^{-\phi}\sigma) - Sb\sigma e^{-\phi}\Big)\Big(\frac{1}{2}(Z_1 - W_1)^2 + \gamma_1^2 \ln(Z_1 + W_1)\Big)$$

$$+\Big(\frac{1}{2(a - e^{\phi/2}b)}\Big)\Big(Sae^{\phi/2}(1 + e^{-\phi}\sigma) + Sb\sigma(e^{-\phi} + \sigma)\Big)$$

$$\Big(\frac{Z_1 - W_1}{2}\Big(ie^{\phi/2}y - \frac{iZ_1e^{\phi/2}y + e^{\phi}b^2}{W_1}\Big)$$

$$+\frac{\gamma_1^2}{2}\Big(\frac{ie^{\phi/2}y + \frac{iZ_1e^{\phi/2}y + e^{\phi}b^2}{W_1}}{Z_1 + W_1}\Big)$$

$$-e^{\phi}b^2 \ln(Z_1 + W_1)\Big) + \Big(\frac{\frac{1}{2}e^{-\phi/2}b}{2(a - e^{-\phi/2}b)^2}\Big)$$

$$\Big(Sae^{-\phi/2}(1 + e^{\phi}\sigma) + Sb\sigma(e^{\phi} + \sigma)\Big)$$

$$\Big(\frac{1}{2}(Z_2 - W_2)^2 + \gamma_2^2 \ln(Z_2 + W_2)\Big) - \Big(\frac{1}{2(a - e^{-\phi/2}b)}\Big)$$

$$\Bigg(-\frac{Sa}{2}e^{-\phi/2}(1+e^{\phi}\sigma)+Sa\,e^{-\phi/2}e^{\phi}\sigma+Sb\sigma e^{\phi}\Bigg)$$

$$\Bigg(\frac{1}{2}(Z_2-W_2)^2+\gamma_2^2\ln(Z_2+W_2)\Bigg)-\Bigg(\frac{1}{2(a-e^{-\phi/2}b)}\Bigg)$$

$$\Bigg(Sa\,e^{-\phi/2}(1+e^{\phi}\sigma)+Sb\sigma(e^{\phi}+\sigma)\Bigg)$$

$$\Bigg(-\frac{Z_2-W_2}{2}\Bigg(i\,e^{-\phi/2}y-\frac{iZ_2e^{-\phi/2}+e^{-\phi}b^2}{W_2}\Bigg)$$

$$-\frac{\gamma_2^2}{2}\Bigg(\frac{i\,e^{-\phi/2}y+\dfrac{iZ_2e^{-\phi/2}y+e^{-\phi}b^2}{W_2}}{Z_2+W_2}\Bigg)$$

$$+e^{-\phi}b^2\ln(Z_2+W_2)\Bigg)\Bigg]\Bigg\}+\frac{Sy^2}{2} \qquad (94)$$

where

$$U = \frac{1}{2} e^{\phi/2}(e^{\phi}+\sigma)(1+e^{-\phi}\sigma) + e^{\phi/2}e^{\phi}(1+e^{-\phi}\sigma)$$
$$+ e^{\phi/2}(e^{\phi}+\sigma)(-e^{-\phi}\sigma)$$

and

$$V = -\frac{1}{2} e^{\phi/2}(1+e^{\phi}\sigma)(e^{-\phi}+\sigma) + e^{-\phi/2}e^{\phi}\sigma(e^{-\phi}+\sigma)$$
$$+ e^{-\phi/2}(1+e^{\phi}\sigma)(-e^{-\phi}).$$

Let $\phi = 0$ and equation (34) takes the form

$$F = R\left\{ \left[\frac{1}{(1+\sigma)(3-\sigma)}\right]\left[\underline{\frac{b}{2(a-b)^2}\left(Sa(1+\sigma)+Sb\sigma(1+\sigma)\right)}\right.\right.$$

$$\left(\frac{1}{2}(z-w)^2+\underline{\gamma^2 \ln(z+w)}\right) + \frac{1}{a-b}\left(\frac{Sa}{2}(1+\sigma)-Sa\sigma\right.$$

$$\left.-Sb\sigma\right)\left(\frac{1}{2}(z-w)^2+\gamma^2 \ln(z+w)\right) + \underline{\left(\frac{1}{a-b}\right)}$$

$$\underline{\left(Sa(1+\sigma)+Sb\sigma(1+\sigma)\right)}\left(\frac{z-w}{2}\left(iY-\frac{iYz+b^2}{w}\right)\right.$$

$$\left.\left.\left.+\frac{a^2-b^2}{2}\left(\frac{iY+\frac{iYz+b^2}{w}}{z+w}\right)\underline{-b^2 \ln(z+w)}\right)\right]\right]\right\}$$

$$+\frac{SY^2}{2}. \tag{95}$$

Equation (95) is the stress function for the effect of a rigid elliptic disk on the stress distribution in an isotropic plate. The problem of a circular disk is the special case when a = b. On setting a = b, parts of equation (95) become indeterminate. However, equation (95) may be evaluated by successive applications of L'Hospital's rule, that is, by differentiating both numerator and denominator with respect to b, remembering that

$$W = \sqrt{Z^2 - a^2 + b^2}.$$

It is suggested that the underlined terms of equation (95) be combined before using L'Hospital's rule. The method is then straightforward and for b = a gives

$$F = R\left\{\left[\frac{1}{(1+\sigma)(3-\sigma)}\right]\left[\frac{S(1+\sigma)^2a^4}{4Z^2} + \frac{Sa^2(1+\sigma)^2}{2}\ln 2Z \right.\right.$$
$$+ Sa^2(1-3\sigma)\ln 2Z - \frac{Sa^4(1+\sigma)^2}{2Z^2}$$
$$\left.\left. + Sa^2(1+\sigma)^2\frac{iY}{Z} + \frac{Sa^4(1+\sigma)^2}{2Z^2}\right]\right\} + SY^2. \tag{96}$$

Since constant terms do not appear in the stresses ln 2Z may be written as ln Z. Also,

$$iY = \frac{Z - \bar{Z}}{2} \tag{97}$$

and noting that two terms of equation (96) cancel, equation (90) reduces to

$$F = R\left\{\frac{S(1+\sigma)a^4}{4(3-\sigma)Z^2} + \frac{S(1-\sigma)a^2}{2(1+\sigma)}\ln Z - \frac{S(1+\sigma)a^2\bar{Z}}{2(3-\sigma)Z}\right\} + \frac{Sy^2}{2}. \qquad (90)$$

Equation (90) is the stress function for the effect of a rigid circular disk on the stress distribution in an isotropic plate. By following the method used in evaluating the constants A and B, it can be shown that u = v = 0 on the boundary of the circular disk.

CHAPTER III

THE EXACT SOLUTION FOR PLANE STRAIN

The Derivation of the Differential Equation for an Orthotropic Plate in a State of Plane Strain

For the problem of plane strain, the dimension of the body in the z direction is considered large, and the body is loaded uniformly in the z direction by forces which act perpendicular to the z-axis. The functions describing these forces are not functions of z. The portion of the body at a distance from the ends has a plane deformation. The displacements u and v are not functions of z and $e_{zz} \equiv w \equiv 0$.

From these assumptions, it follows that equations (1), (2), and (4) become

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E_x}\left(1 - \sigma_{zx}\sigma_{xz}\right)X_x - \frac{1}{E_y}\left(\sigma_{yx} + \sigma_{zx}\sigma_{yz}\right)Y_y \tag{99}$$

$$e_{yy} = \frac{\partial v}{\partial y} = -\frac{1}{E_x}\left(\sigma_{xy} + \sigma_{zy}\sigma_{xz}\right)X_x + \frac{1}{E_y}\left(1 - \sigma_{zy}\sigma_{yz}\right)Y_y \tag{100}$$

$$e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{\mu_{xy}} X_y . \qquad (101)$$

Now,

$$\frac{\sigma_{zy}}{E_z} = \frac{\sigma_{yz}}{E_y}, \quad \frac{\sigma_{xz}}{E_x} = \frac{\sigma_{zx}}{E_z}, \quad \frac{\sigma_{yx}}{E_y} = \frac{\sigma_{xy}}{E_x}.^{1} \qquad (102)$$

It follows from equations (12), (13), (99), (100), (101), and (102) that

$$\frac{\partial^4 F}{\partial x^4} + \frac{E_y}{(1-\sigma_{zy}\sigma_{yz})}\left(\frac{1}{\mu_{xy}} - \frac{2(\sigma_{xy}+\sigma_{xz}\sigma_{zy})}{E_x}\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{E_y(1-\sigma_{zx}\sigma_{xz})}{E_x(1-\sigma_{zy}\sigma_{yz})}\frac{\partial^4 F}{\partial y^4} = 0. \qquad (103)$$

Let $\eta = \epsilon y$ and

$$\epsilon = \sqrt[4]{\frac{E_x(1-\sigma_{zy}\sigma_{yz})}{E_y(1-\sigma_{zx}\sigma_{xz})}} . \qquad (104)$$

Equation (103) may now be written

$$\frac{\partial^4 F}{\partial x^4} + 2K'\frac{\partial^4 F}{\partial x^2 \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = 0 \qquad (105)$$

where

$$K' = \frac{1}{2}\sqrt{\frac{E_x E_y}{(1-\sigma_{zx}\sigma_{xz})(1-\sigma_{zy}\sigma_{yz})}}\left(\frac{1}{\mu_{xy}} - \frac{2(\sigma_{xy}+\sigma_{xz}\sigma_{zy})}{E_x}\right). \qquad (106)$$

---

[1] March, Stress-Strain Relations in Wood and Plywood, p. 7.

The general solution of equation (105) is equation (35) where ϵ and η are defined by equation (104) and α and β are defined by equations (29) and (30) with K' replacing K.

A' and B' Determined from the Interior Boundary Conditions

The boundary conditions for the problem in plane strain are the same as those of the problem solved in Chapter II for plane stress.

A suitable stress function is again equation (36) where ϵ, η, α, β, and K' are defined in the preceding section, and the constants $A' = A_1' + iA_2'$ and $B' = B_1' + iB_2'$ are primed to distinguish them from the constants of Chapter II. This new stress function is designated equation (36'). Equations (47), (48), and (49) with A and B primed are designated equations (47'), (48'), and (49'), the stresses Y_y, X_x, and X_y, respectively.

The result of substituting equations (47') and (48') into equation (99) and integrating with respect to x is

$$U = R\left\{-A'\left[\frac{\alpha^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})}{E_x} + \frac{\sigma_{yx}+\sigma_{zx}\sigma_{yz}}{E_y}\right]\left[\frac{1}{Z_1+W_1}\right] - B'\left[\frac{\beta^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})}{E_x} + \frac{\sigma_{yx}+\sigma_{zx}\sigma_{yz}}{E_y}\right]\left[\frac{1}{Z_2+W_2}\right]\right\} + \frac{Sx}{E_x}(1-\sigma_{zx}\sigma_{xz}) + U_0(Y). \qquad (107)$$

$U_0(y)$, an arbitrary function of y, is identically zero. This can be shown in a manner analogous to the method used in establishing $u_0(y) \equiv 0$.

Since $u = 0$ on the boundary of the rigid elliptic disk, the result of substituting from equations (75) and (76) (α, β, and ϵ defined as in Chapter III) is

$$
\begin{aligned}
u = \Bigg[& \frac{-A_1'}{a+\alpha\epsilon b}\left(\frac{\alpha^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})}{E_x} + \frac{\sigma_{yx}+\sigma_{zx}\sigma_{yz}}{E_y}\right) \\
& - \frac{B_1'}{a+\beta\epsilon b}\left(\frac{\beta^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})}{E_x} + \frac{\sigma_{yx}+\sigma_{zx}\sigma_{yz}}{E_y}\right) \\
& + \frac{Sa}{E_x}(1-\sigma_{zx}\sigma_{xz})\Bigg]\cos\theta - \Bigg[\left(\frac{A_2'}{a+\alpha\epsilon b}\right) \\
& \left(\frac{\alpha^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})}{E_x} + \frac{\sigma_{yx}+\sigma_{zx}\sigma_{yz}}{E_y}\right) + \left(\frac{B_2'}{a+\beta\epsilon b}\right) \\
& \left(\frac{\beta^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})}{E_x} + \frac{\sigma_{yx}+\sigma_{zx}\sigma_{yz}}{E_y}\right)\Bigg]\sin\theta = 0. \qquad (108)
\end{aligned}
$$

Since $u = 0$ for $0 \leqq \theta \leqq 2\pi$, it follows that the terms in the square brackets may be equated to zero. Equating these to zero and simplifying by virtue of equation (102) leads to

$$
\begin{aligned}
& A_1'\left[\frac{\alpha^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})+\sigma_{xy}+\sigma_{xz}\sigma_{zy}}{a+\alpha\epsilon b}\right] \\
& + B_1'\left[\frac{\beta^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})+\sigma_{xy}+\sigma_{xz}\sigma_{zy}}{a+\beta\epsilon b}\right]
\end{aligned}
$$

$$-Sa(1-\sigma_{zx}\sigma_{xz})=0 \tag{109}$$

and

$$\frac{A_2'}{a+\alpha\epsilon b}\left[\frac{\alpha^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})}{E_x}+\frac{\sigma_{yx}+\sigma_{zx}\sigma_{yz}}{E_y}\right]$$

$$+\frac{B_2'}{a+\beta\epsilon b}\left[\frac{\beta^2\epsilon^2(1-\sigma_{zx}\sigma_{xz})}{E_x}+\frac{\sigma_{yx}+\sigma_{zx}\sigma_{yz}}{E_y}\right]=0. \tag{110}$$

The result of substituting equations (47') and (48') into equation (100) and integrating with respect to y is

$$V=R\left\{A'\left[\frac{\alpha^2\epsilon^2(\sigma_{xy}+\sigma_{zy}\sigma_{xz})}{E_x}+\frac{1-\sigma_{zy}\sigma_{yz}}{E_y}\right]\frac{1}{i\alpha\epsilon(Z_1+W_1)}\right.$$

$$\left.+B'\left[\frac{\beta^2\epsilon^2(\sigma_{xy}+\sigma_{zy}\sigma_{xz})}{E_x}+\frac{1-\sigma_{zy}\sigma_{yz}}{E_y}\right]\frac{1}{i\beta\epsilon(Z_2+W_2)}\right\}$$

$$-\frac{SY}{E_x}(\sigma_{xy}+\sigma_{zy}\sigma_{xz})+V_0(x). \tag{111}$$

$V_0(x)$, an arbitrary function of x, is identically zero. This can be shown in a manner analogous to the method used in establishing $v_0(x)$ identically zero.

Since v = 0 on the boundary of the rigid elliptic disk, the result of substituting from equations (75) and (76) (α, β, and ϵ defined as in Chapter III) is

$$V=\left[\frac{-A_1'}{\alpha\epsilon(a+\alpha\epsilon b)}\left(\frac{\alpha^2\epsilon^2(\sigma_{xy}+\sigma_{zy}\sigma_{xz})}{E_x}+\frac{1-\sigma_{zy}\sigma_{yz}}{E_y}\right)\right.$$

$$-\frac{B_1'}{\beta\epsilon(a+\beta\epsilon b)}\left(\frac{\beta^2\epsilon^2(\sigma_{xy}+\sigma_{zy}\sigma_{xz})}{E_x}+\frac{1-\sigma_{zy}\sigma_{yz}}{E_y}\right)$$

$$-\frac{Sb}{E_x}(\sigma_{xy}+\sigma_{zy}\sigma_{xz})\Bigg]\sin\theta+\Bigg[\left(\frac{A_2'}{\alpha\epsilon(a+\alpha\epsilon b)}\right)$$

$$\left(\frac{\alpha^2\epsilon^2(\sigma_{xy}+\sigma_{zy}\sigma_{xz})}{E_x}+\frac{1-\sigma_{zy}\sigma_{yz}}{E_y}\right)+\left(\frac{B_2'}{\beta\epsilon(a+\beta\epsilon b)}\right)$$

$$\left(\frac{\beta^2\epsilon^2(\sigma_{xy}+\sigma_{zy}\sigma_{xz})}{E_x}+\frac{1-\sigma_{zy}\sigma_{yz}}{E_y}\right)\Bigg]\cos\theta=0. \qquad (112)$$

Since v = 0 for $0 \leqq \theta \leqq 2\pi$, it follows that the terms in the square brackets may be equated to zero. Equating these to zero and simplifying by virtue of equations (102) leads to

$$A_1'\left[\frac{\alpha^2\epsilon^2\sigma_{yx}(\sigma_{xy}+\sigma_{zy}\sigma_{xz})+\sigma_{xy}(1-\sigma_{zy}\sigma_{yz})}{\alpha\epsilon(a+\alpha\epsilon b)}\right]$$

$$+B_1'\left[\frac{\beta^2\epsilon^2\sigma_{yx}(\sigma_{xy}+\sigma_{zy}\sigma_{xz})+\sigma_{xy}(1-\sigma_{zy}\sigma_{yz})}{\beta\epsilon(a+\beta\epsilon b)}\right]$$

$$+Sb\,\sigma_{yx}(\sigma_{xy}+\sigma_{zy}\sigma_{xz})=0 \qquad (113)$$

and

$$\frac{A_2'}{\alpha\epsilon(a+\alpha\epsilon b)}\left[\frac{\alpha^2\epsilon^2(\sigma_{xy}+\sigma_{zy}\sigma_{xz})}{E_x}+\frac{1-\sigma_{zy}\sigma_{yz}}{E_y}\right]$$

$$+\frac{B_2'}{\beta\epsilon(a+\beta\epsilon b)}\left[\frac{\beta^2\epsilon^2(\sigma_{xy}+\sigma_{zy}\sigma_{xz})}{E_x}+\frac{1-\sigma_{zy}\sigma_{yz}}{E_y}\right]=0. \qquad (114)$$

Solving equations (110) and (114) simultaneously gives

$$A_2' = B_2' = 0. \tag{115}$$

Solving equations (109) and (113) simultaneously gives

$$A_1' = \frac{A_{11} + A_{12}}{A_{21} - A_{22}} \tag{116}$$

and

$$B_1' = -\frac{B_{11} + B_{12}}{B_{21} - B_{22}} \tag{117}$$

where

$$A_{11} = \frac{Sa\alpha}{\epsilon}(1 - \sigma_{zx}\sigma_{xz})\left[\beta^2\epsilon^2\sigma_{yx}(\sigma_{xy} + \sigma_{zy}\sigma_{xz}) + \sigma_{xy}(1 - \sigma_{zy}\sigma_{yz})\right],$$

$$A_{12} = Sb\sigma_{yx}(\sigma_{xy} + \sigma_{zy}\sigma_{xz})\left[\beta^2\epsilon^2(1 - \sigma_{zx}\sigma_{xz}) + \sigma_{xy} + \sigma_{xz}\sigma_{zy}\right],$$

$$A_{21} = \alpha\left[\frac{\alpha^2\epsilon^2(1 - \sigma_{zx}\sigma_{xz}) + \sigma_{xy} + \sigma_{xz}\sigma_{zy}}{\epsilon(a + \alpha\epsilon b)}\right]$$

$$\left[\beta^2\epsilon^2\sigma_{yx}(\sigma_{xy} + \sigma_{zy}\sigma_{xz}) + \sigma_{xy}(1 - \sigma_{zy}\sigma_{yz})\right],$$

$$A_{22} = \beta\left[\frac{\alpha^2\epsilon^2\sigma_{yx}(\sigma_{xy} + \sigma_{zy}\sigma_{xz}) + \sigma_{xy}(1 - \sigma_{zy}\sigma_{yz})}{\epsilon(a + \alpha\epsilon b)}\right]$$

$$\left[\beta^2\epsilon^2(1 - \sigma_{zx}\sigma_{xz}) + \sigma_{xy} + \sigma_{xz}\sigma_{zy}\right],$$

$$B_{11} = \frac{Sa\beta}{\epsilon}(1 - \sigma_{zx}\sigma_{xz})\left[\alpha^2\epsilon^2\sigma_{yx}(\sigma_{xy} + \sigma_{zy}\sigma_{xz}) + \sigma_{xy}(1 - \sigma_{zy}\sigma_{yz})\right],$$

$$B_{12} = S b \sigma_{yx}(\sigma_{xy} + \sigma_{zy}\sigma_{xz})\left[\alpha^2\epsilon^2(1 - \sigma_{zx}\sigma_{xz}) + \sigma_{xy} + \sigma_{xz}\sigma_{zy}\right],$$

$$B_{21} = \alpha\left[\frac{\alpha^2\epsilon^2(1 - \sigma_{zx}\sigma_{xz}) + \sigma_{xy} + \sigma_{xz}\sigma_{zy}}{\epsilon(a + \beta\epsilon b)}\right]$$

$$\left[\beta^2\epsilon^2\sigma_{yx}(\sigma_{xy} + \sigma_{zy}\sigma_{xz}) + \sigma_{xy}(1 - \sigma_{zy}\sigma_{yz})\right],$$

and

$$B_{22} = \beta\left[\frac{\alpha^2\epsilon^2\sigma_{yx}(\sigma_{xy} + \sigma_{zy}\sigma_{xz}) + \sigma_{xy}(1 - \sigma_{zy}\sigma_{yz})}{\epsilon(a + \beta\epsilon b)}\right]$$

$$\left[\beta^2\epsilon^2(1 - \sigma_{zx}\sigma_{xz}) + \sigma_{xy} + \sigma_{xz}\sigma_{zy}\right].$$

Equation (36') with B_1', B_2', A_1', and A_2' defined by equations (115), (116), and (117) is the stress function giving the effect of a rigid elliptic cylinder on the stress distribution in a body of orthotropic material in a state of plane strain.

The Isotropic Stress Function as the Limiting Case of the Orthotropic Stress Function

To obtain the stress function for the isotropic case, first set $\epsilon = 1$, and allow $\phi \longrightarrow 0$. It follows that

$$F = R\left\{\left[\frac{1}{\sigma(1+\sigma)^2(3-4\sigma)}\right]\left[\left(\frac{Sa}{2}\sigma(1-\sigma^2)(1+\sigma)\right.\right.\right.$$

$$-Sa\sigma^2(1-\sigma^2)(1+\sigma)-Sb\sigma^2(1-\sigma^2)(1+\sigma)\Big)\Big(\frac{1}{a-b}\Big)$$

$$\Big(\frac{1}{2}(Z-W)^2+(a^2-b^2)\ln(Z+W)\Big)\Big(Sa\sigma(1-\sigma^2)(1+\sigma)$$

$$+Sb\sigma^2(1+\sigma)^2\Big)\Big(\frac{b}{2(a-b)^2}\Big)\Big(\frac{1}{2}(Z-W)^2+(a^2-b^2)\ln(Z+W)\Big)$$

$$+\Big(Sa\sigma(1-\sigma^2)(1+\sigma)+Sb\sigma^2(1+\sigma)^2\Big)\Big(\frac{1}{a-b}\Big)$$

$$\Big(\frac{Z-W}{2}\Big(iY-\frac{ZiY+b^2}{W}\Big)+\frac{(a^2-b^2)}{2}\Big(\frac{iY+\frac{ZiY+b^2}{W}}{Z+W}\Big)$$

$$-b^2\ln(Z+W)\Big)\Big]\Big\}+\frac{SY^2}{2}. \qquad (118)$$

In order to obtain the stress function for a rigid circular cylinder, set b = a. Equation (118) evaluated becomes

$$F=R\Big\{\Big[\frac{1}{\sigma(1+\sigma)^2(3-4\sigma)}\Big]\Big[Sa^2\sigma(1-\sigma^2)(1+\sigma)(1-4\sigma)\ln 2Z$$

$$+\frac{Sa^4\sigma(1+\sigma)^2}{4Z^2}+\frac{Sa^2}{2}\sigma(1+\sigma)^2\ln 2Z-\frac{Sa^4\sigma(1+\sigma)^2}{2Z^2}$$

$$+Sa^2\sigma(1+\sigma)^2\Big(\frac{iY}{Z}+\frac{a^2}{2Z^2}\Big)\Big]\Big\}+\frac{SY^2}{2}. \qquad (119)$$

Now,

$$iY=\frac{Z-\bar{Z}}{2} \quad \text{and} \quad Y^2=R\Big\{\frac{1}{2}(\bar{Z}Z-Z^2)\Big\}.$$

Equation (119) may now be written, dropping a constant term which does not appear in the stresses,

$$F = R\left\{\frac{Sa^2}{2}(1-2\sigma)\ln z + \frac{Sa^4}{4(3-4\sigma)z^2} - \frac{Sa^2}{2(3-4\sigma)}\frac{\bar{z}}{z} + \frac{S}{4}\bar{z}z - \frac{S}{4}z^2\right\}. \tag{120}$$

It is interesting to note that using a different method Professor I. S. Sokolnikoff[2] obtained this same stress function.

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[2] Sokolnikoff, op. cit., chap. v.

CHAPTER IV

THE PLANE STRESS SOLUTION APPLIED TO A SITKA SPRUCE PLATE

The results of Chapter II are applied to a large plain-sawn rectangular plate of Sitka spruce containing a rigid elliptic disk at its center. It is possible to apply the results to a large finite plate since the stress concentration is localized near the rigid disk. The stresses approach the values of the stresses at infinity at a distance of only a few multiples of the major axis of the elliptic disk. Figure 4 illustrates a plain-sawn wooden plate with its axes of symmetry. Here the x- and y-axes are parallel to the L and T axes, respectively. The origin is at the center of the ellipse with the axes of the ellipse coincident with the coordinate axes (see Figure 3).

The stresses at any point in the plate are given by equations (47), (48), and (49) where A equal A_1 is given by equation (88) and B equal B_1 is given by equation (89).

Shear stresses are probably most important in producing failure in a wooden plate. For this reason a formula for the shear stress along the boundary of the rigid disk is useful. The shear stress given by equation (49) evaluated on the

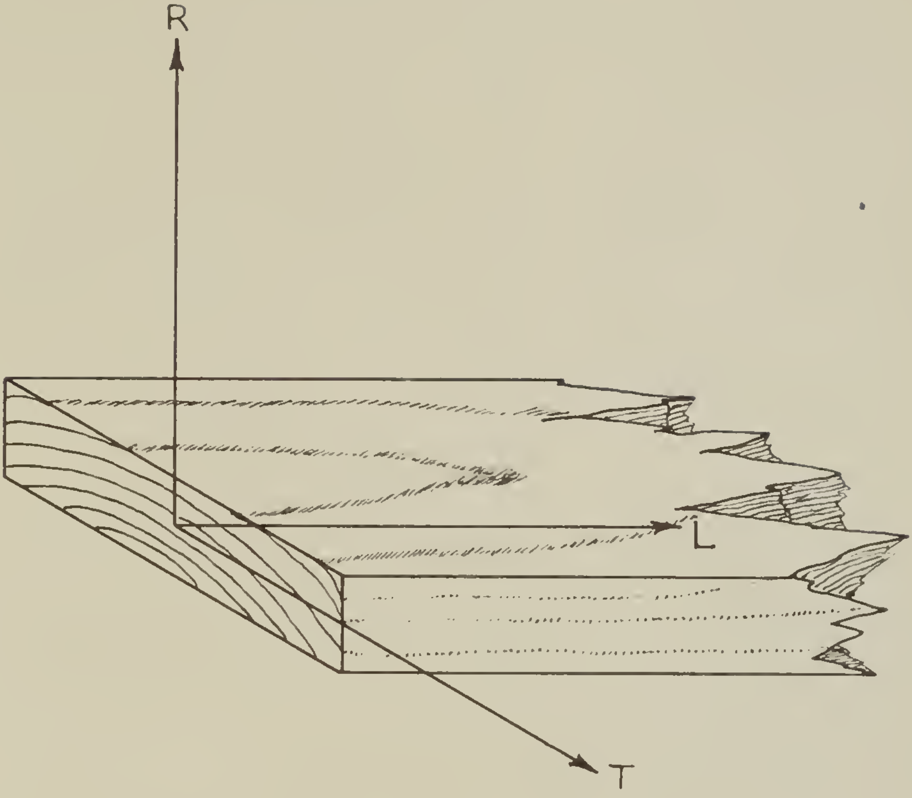

Figure 4.--Plain-sawn wooden plate with axes of elastic symmetry.

boundary of the rigid disk reduces with the use of equations (73), (74), (75), and (76) to

$$X_y\Big|_{0 \leqq \theta \leqq \frac{\pi}{2}} = \frac{\alpha \epsilon A_1 \sin\theta \cos\theta}{\alpha^2 \epsilon^2 b^2 \cos^2\theta + a^2 \sin^2\theta}$$

$$+ \frac{\beta \epsilon B_1 \sin\theta \cos\theta}{\beta^2 \epsilon^2 b^2 \cos^2\theta + a^2 \sin^2\theta} \qquad (121)$$

where A_1 and B_1 are determined by equations (38) and (39).

For a ratio of a to b equal to or greater than one, the normal stress X_x has a relatively high value at the point (a, 0). From equations (40), (38), and (39), we have

$$X_x\Big|_{\substack{x=a\\y=0}} = \frac{N_1 - N_2}{D} + S \qquad (122)$$

where

$$N_1 = Sa\alpha^2\epsilon(1+\epsilon^2\beta^2\sigma_{yx}) + S\alpha\epsilon^2 b\sigma_{yx}(\epsilon^2\beta^2+\sigma_{xy}),$$

$$N_2 = Sa\beta^2\epsilon(1+\epsilon^2\alpha^2\sigma_{yx}) + S\beta\epsilon^2 b\sigma_{yx}(\epsilon^2\alpha^2+\sigma_{xy}),$$

$$D = \alpha b(\epsilon^2\alpha^2+\sigma_{xy})(1+\epsilon^2\beta^2\sigma_{yx}) - \beta b(1+\epsilon^2\alpha^2\sigma_{yx})(\epsilon^2\beta^2+\sigma_{xy}).$$

The values of E_L, E_T, σ_{LT}, and μ_{LT}, that is, E_x, E_y, σ_{xy}, and μ_{xy}, respectively, for Sitka spruce are tabulated in Table 1.[1] We obtain σ_{yx} from equation (17). In addition, S, a, b, ϵ, α, and β are required to obtain the value of

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[1] All tables and figures (except Figures 3 and 4) mentioned in this chapter are grouped in the Appendix.

A_1 and B_1. S is the uniform tension applied in the direction of the grain, and a and b determine the ellipse. The values of α, β, and ϵ are computed by means of equations (20), (29), (30), and (22). Constants evaluated for Sitka spruce occurring in the expressions for A_1 and B_1 are tabulated in Table II. Values of A_1 and B_1 for several ratios of a to b are given in Table III.

The ratios of the stresses computed for various points in a plate containing a rigid circular disk to the normal stress at infinity are given in Tables IV to VII. These ratios and the ratios computed for the same problem with an isotropic plate, where σ, Poisson's ratio, is taken to be 0.3, are graphed in Figures 5 to 10.

The ratios of the stresses computed for points in a Sitka spruce plate containing a rigid elliptic disk ($b = 5a$) to the normal stress at infinity are given in Tables VIII to X. These ratios are graphed in Figures 11 to 15.

The ratios of the stresses computed for various points in a plate containing a rigid elliptic disk ($a = 5b$) to the normal stress at infinity are given in Tables XI and XII. These ratios are graphed in Figures 16 and 17.

The ratio of the shear stress computed for points along the boundary of a rigid elliptic disk ($a = 10b$) to the normal stress at infinity are given in Table XIII. These ratios are graphed in Figure 18. We find from equation (122) that the normal stress X_x is 3.4443S at the point (x = a = 10b, 0).

It is interesting to compare the stresses in a Sitka spruce plate containing a circular hole of radius a obtained by Professor Smith and the stresses in a Sitka spruce plate containing a rigid circular disk of radius a obtained here. Throughout a plate containing a circular hole $|Y_y|$ is less than 0.22S and throughout a plate containing a rigid circular disk $|Y_y|$ is less than 0.03S. The maximum normal stress for a plate containing a circular hole was found to be X_x equal 5.84S at the point (0,a) while for a plate containing a rigid circular disk the maximum normal stress was X_x equal 1.23S at the point (a,0). The maximum normal stress in the latter plate is smaller and located at a different point than in the former plate. In fact, X_x is practically zero at the point (0,a) for the plate containing the rigid circular disk. Comparison of the points at which the maximum shear stresses occur does not show such a difference in position. The maximum shear occurs along the boundary of the disk and the edge of the hole at about 77° and 78°, respectively. At the edge of the hole the maximum shear stress is larger than the maximum shear stress on the boundary of the disk. The maximum shear stress on the edge of the hole is 0.71S and on the boundary of the disk it is 0.16S.

From these results, it is probable that in a plate containing a circular disk or a plate containing a circular hole, for S sufficiently large, failure in the plate will occur approximately along the line $y = \pm a \sin 77\frac{1}{2}^\circ = \pm 0.976a$ with the crack beginning at the disk or hole. However, higher values of S will probably be required to produce failure in the plate containing the rigid circular disk than will be required to produce failure in the plate containing the circular hole.

It should be noted that the normal stress X_x computed for the point (a,0) increases as the ratio of a to b increases. For the ratio of a to b equal one-fifth, X_x at the point (a,0) is 1.03S and for the ratio a to b equal ten, X_x at the point (a,0) is 3.44S. The absolute value of the maximum shear stress on the boundary of the disk increases as the ratio of a to b increases, and for the ratio of a to b equal to or greater than one, the maximum shear stress occurs at smaller values of θ as the ratio increases. For the ratio a to b equal one-fifth, the maximum shear stress is approximately -0.04S at θ equal 70°, while for the ratio a to b equal ten, the maximum shear stress is about 0.44S at θ equal approximately 24°.

On the basis of this analysis, we conclude that a large plain-sawn Sitka spruce plate containing a small rigid elliptic disk, such as a knot or a steel plug, is probably weaker than a homogeneous plate but stronger than a plate containing an elliptic hole of the same size. Further, the orientation of the major axis is important. We will expect a plate containing a rigid elliptic disk with the major axis perpendicular to the direction of the grain to withstand higher values of S, where

S is the uniform tension in the direction of the grain applied at the edges perpendicular to the grain, without failure than a plate containing a rigid elliptic disk of the same size with its major axis parallel to the direction of the grain. The ratio of a to b is important. As the ratio increases the maximum normal stress and the absolute value of the maximum shear stress increase. Therefore, we expect a large plate containing a rigid elliptic disk with the major axis parallel to the grain to withstand higher values of S without failure than a similar plate where the ratio of a to b is greater.

APPENDIX

TABLE I

ELASTIC MODULI OF SITKA SPRUCE*

σ_{LT}	0.464				
E_L	1,679,000	lbs.	per	sq.	in.
E_T	76,000	"	"	"	"
μ_{LT}	112,000	"	"	"	"

*The elastic moduli of Sitka spruce are average values obtained from a large number of tests made at the U. S. Forest Products Laboratory.

TABLE II

CONSTANTS EVALUATED FOR SITKA SPRUCE*

σ_{yx}	0.0210	α/ϵ	0.7449
ϵ	2.168	$\beta\epsilon$	1.342
ϵ^2	4.700	$\beta^2\epsilon^2$	1.802
K	1.496	β/ϵ	0.2856
α	1.615	$1+\epsilon^2\alpha^2\sigma_{yx}$	1.257
α^2	2.608	$\epsilon^2\alpha^2+\sigma_{yx}$	12.72
β	0.6192	$1+\alpha\epsilon$	4.501
β^2	0.3834	$1+\epsilon^2\beta^2\sigma_{yx}$	1.0378
$\alpha\epsilon$	3.501	$\epsilon^2\beta^2+\sigma_{xy}$	2.266
$\alpha^2\epsilon^2$	12.26	$1+\beta\epsilon$	2.342

*The x and y axes are taken along the L and T axes of symmetry, respectively.

TABLE III

THE CONSTANTS A_1 AND B_1 EVALUATED FOR SITKA SPRUCE

Ellipse	A_1	B_1
b = a	[illegible]	[illegible]
b = 5a	$6.076Sa^2$	$-1.450Sa^2$
a = 5b	$3.3628Sb^2$	$-1.450Sb^2$
a = 10b	$11.04Sb^2$	$-4.851Sb^2$

TABLE IV

VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE X-AXIS EXTERIOR TO A RIGID CIRCULAR DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

x	$\frac{X_x}{S}$	$\frac{Y_y}{S}$
1a	1.2213	+[illegible]
2a	1.1357	-0.0001
3a	1.1074	-0.0237
4a	1.0995	-0.0256
5a	1.0699	-0.0273

TABLE V

VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE y-AXIS EXTERIOR TO A RIGID CIRCULAR DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

y	$\frac{X_x}{S}$	$\frac{Y_y}{S}$
1.00a	0.010	+0.0216
1.10a	0.7295	
1.25a	0.8180	
1.50a	0.9089	-0.0033
2.00a	0.9601	-0.0072
3.00a	0.9847	-0.0032
4.00a	0.9918	-0.0018
5.00a	0.9949	-0.0011

TABLE VI

VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE BOUNDARY OF A RIGID CIRCULAR DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

θ	$\frac{X_y}{S}$
0°	0
10°	−0.0004
20°	+0.0010
30°	+0.0068
40°	+0.0198
50°	+0.0442
60°	+0.0841
70°	+0.1348
75°	+0.1525
80°	+0.1467
85°	+0.0959
90°	0

TABLE VII

VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE LINE $x = 2a$ EXTERIOR TO A RIGID CIRCULAR DISK OF RADIUS a WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

y	$\frac{X_y}{S}$
0	0
0.5000a	+0.0279
0.7500a	+0.0370
0.9659a	+0.0348
1.0000a	+0.0334
1.5000a	+0.0090
2.0000a	−0.0001
3.0000a	−0.0018

TABLE VIII

VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE x-AXIS EXTERIOR TO A RIGID ELLIPTIC DISK ($b = 5a$) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

x	$\frac{X_x}{S}$	$\frac{Y_y}{S}$
1a	1.0280	0.0216
2a	1.0316	0.0174
3a	1.0343	0.0137
4a	1.0360	0.0107
5a	1.0368	0.0082

TABLE IX

VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE y-AXIS EXTERIOR TO A RIGID ELLIPTIC DISK (b = 5a) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

y	$\frac{X_x}{S}$	$\frac{Y_y}{S}$
5.00a	-0.0357	-0.0759
5.05a	+0.7413	
5.10a	+0.8594	
5.25a	+0.9309	
5.50a	+0.9611	
5.60a	+0.9666	
6.00a	+0.9786	-0.0198
7.00a	+0.9890	-0.0107
8.00a	+0.9928	-0.0071
9.00a	+0.9949	
10.00a	+0.9954	

TABLE X

VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE BOUNDARY OF A RIGID ELLIPTIC DISK (b = 5a) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

θ	$\frac{X_y}{S}$
0°	0
10°	-0.0034
20°	-0.0071
30°	-0.0111
40°	-0.0159
50°	-0.0218
60°	-0.0295
70°	-0.0381
80°	-0.0213
90°	0

TABLE XI

VALUES OF THE NORMAL STRESS COMPONENTS AT POINTS ALONG THE x-AXIS EXTERIOR TO A RIGID ELLIPTIC DISK (a = 5b) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

x	$\frac{X_x}{S}$
1a	2.2133
2a	1.2340
3a	1.0987
4a	1.0545
5a	1.0344

TABLE XII

VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE BOUNDARY OF A RIGID ELLIPTIC DISK (a = 5b) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

θ	$\frac{X_y}{S}$
0°	0
5°	0.0053
10°	0.0417
20°	0.1632
30°	0.2511
40°	0.2790
50°	0.2600
60°	0.2124
70°	0.1485
80°	0.0760
90°	0

TABLE XIII

VALUES OF THE SHEAR STRESS COMPONENT X_y AT POINTS ALONG THE BOUNDARY OF A RIGID ELLIPTIC DISK (a = 10b) WITH CENTER AT THE ORIGIN FOR A PLAIN-SAWN PLATE OF SITKA SPRUCE

θ	$\frac{X_y}{S}$
0°	0
5°	0.0513
10°	0.2337
15°	0.3674
20°	0.4240
25°	0.4305
30°	0.4089
40°	0.3381
50°	0.2606
60°	0.1896
70°	0.1224
80°	0.0601
90°	0

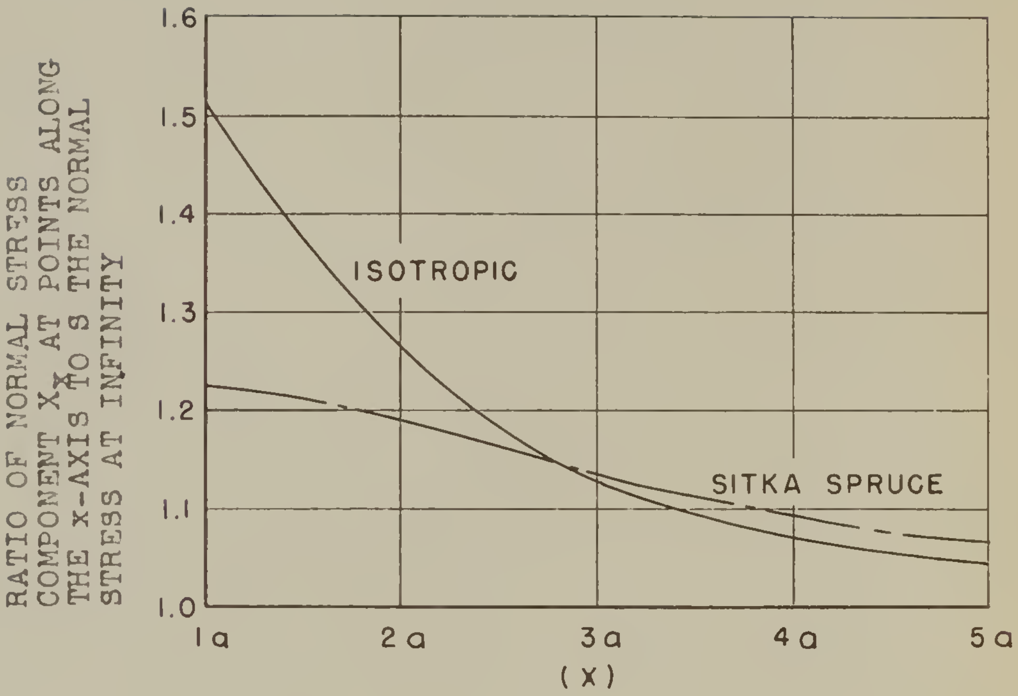

Figure 5.-- Variation of the normal stress component X_x at points along the x-axis exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

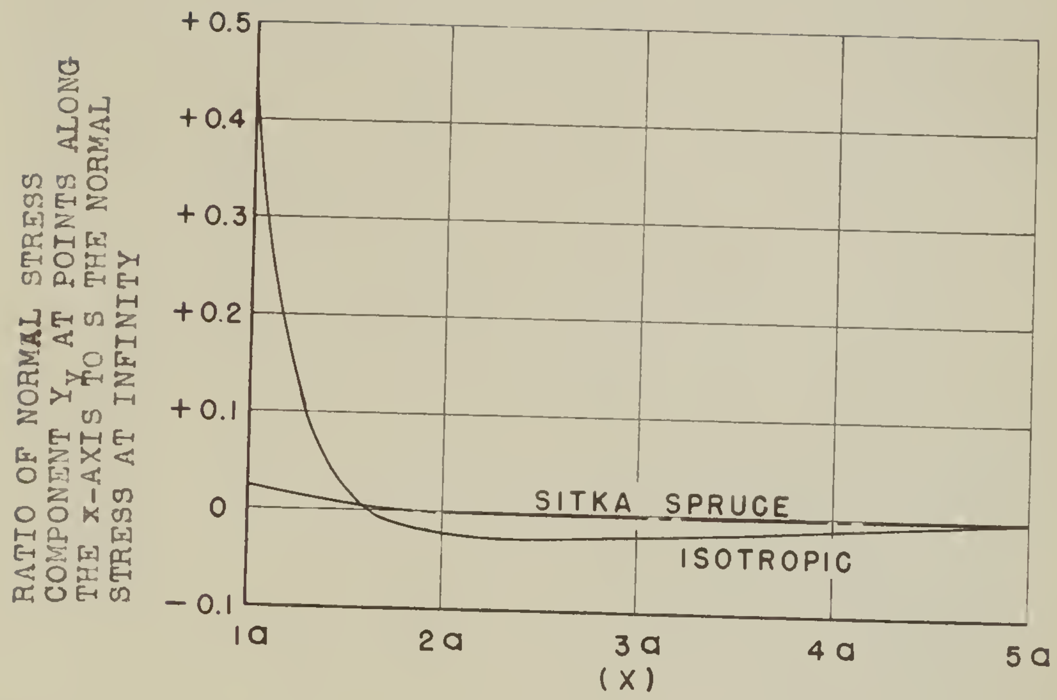

Figure 6.--Variation of the normal stress component Y_y at points along the x-axis exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

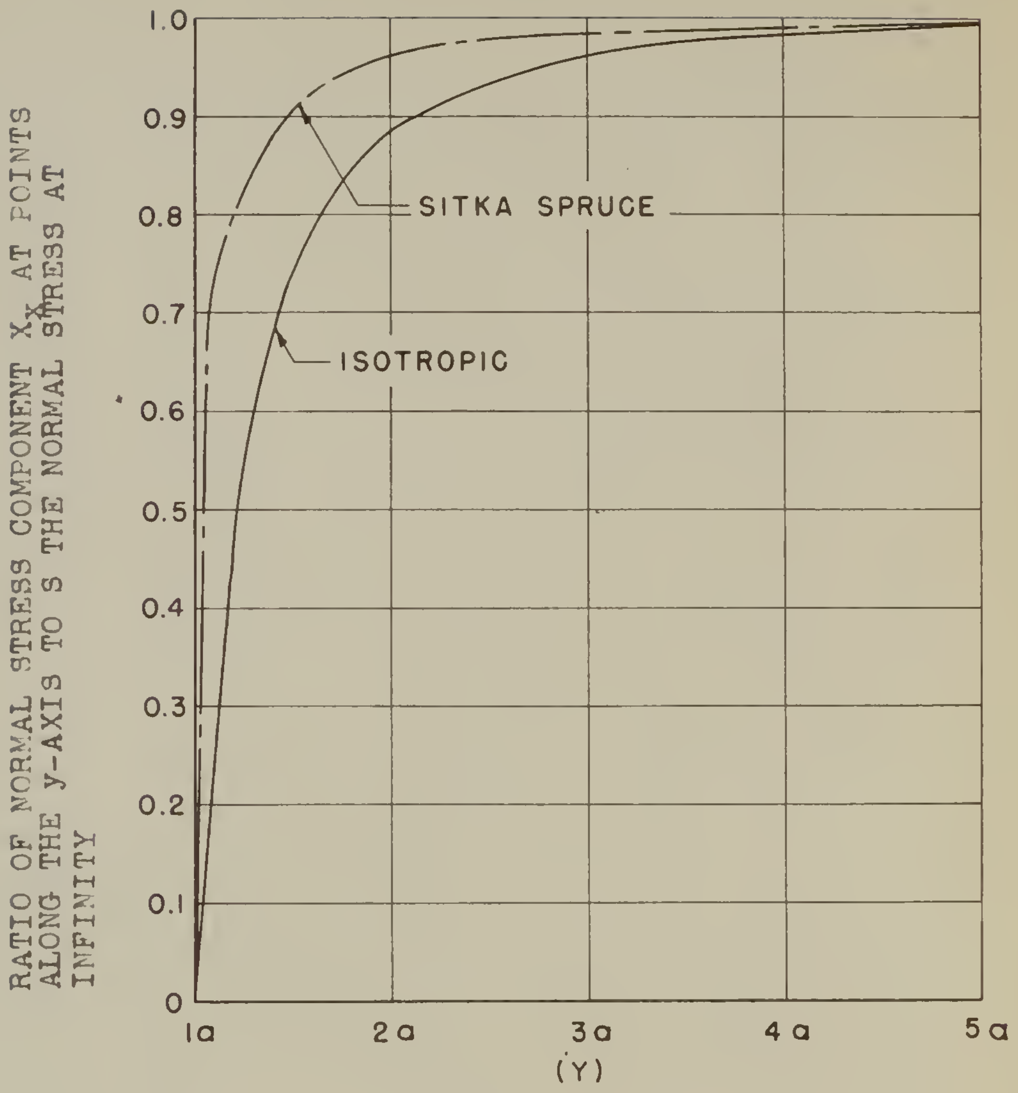

Figure 7.--Variation of the normal stress component X_x at points along the y-axis exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

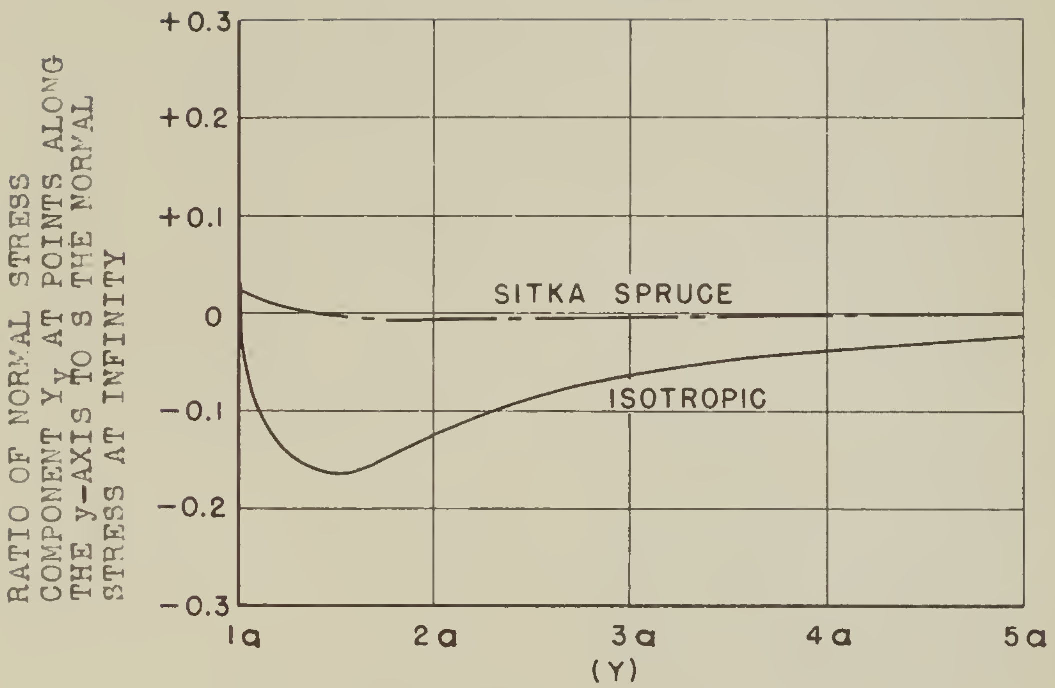

Figure 8.--Variation of the normal stress component Y_y at points along the y-axis exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

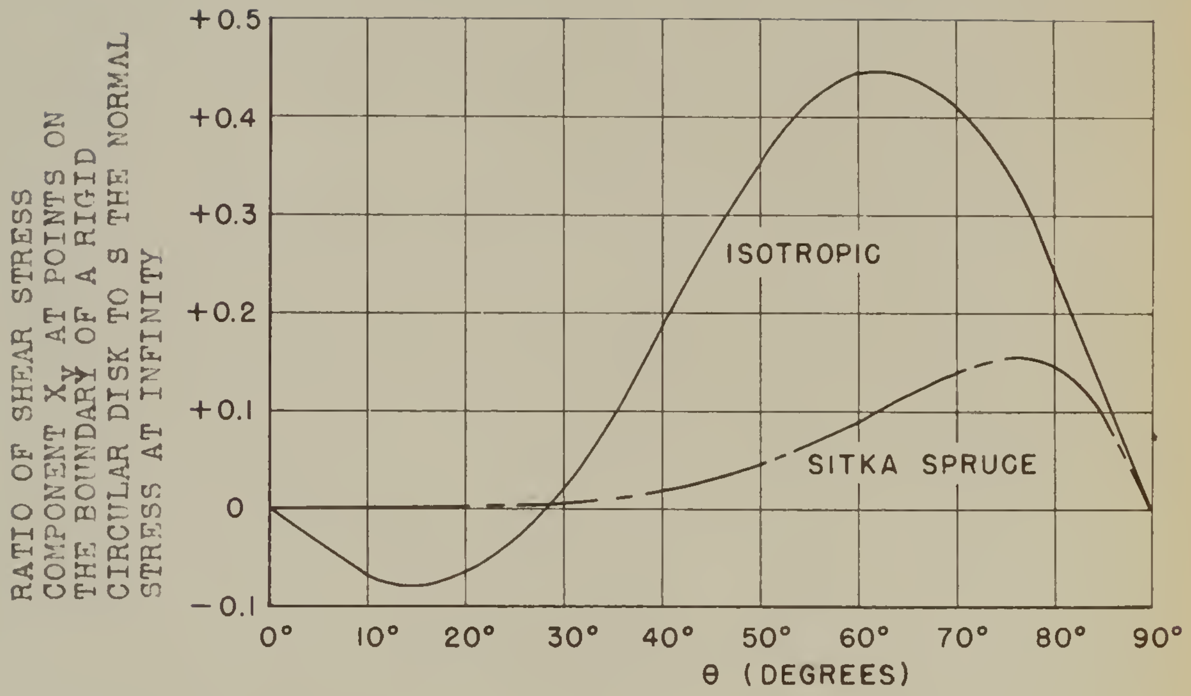

Figure 9.--Variation of the shear stress component X_y at points along the boundary of a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

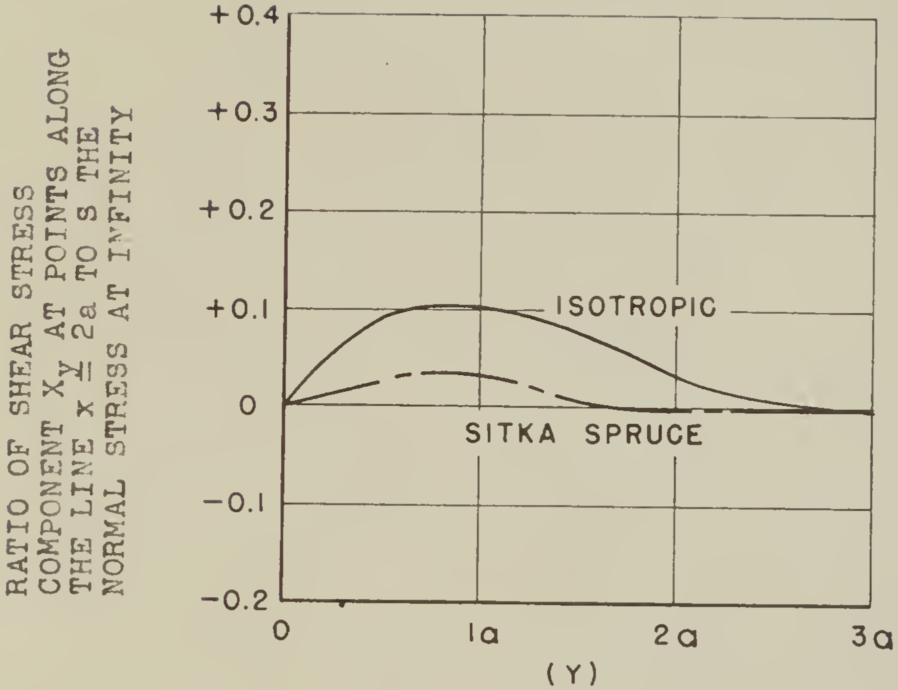

Figure 10.--Variation of the shear stress component X_y at points along the line x = 2a exterior to a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

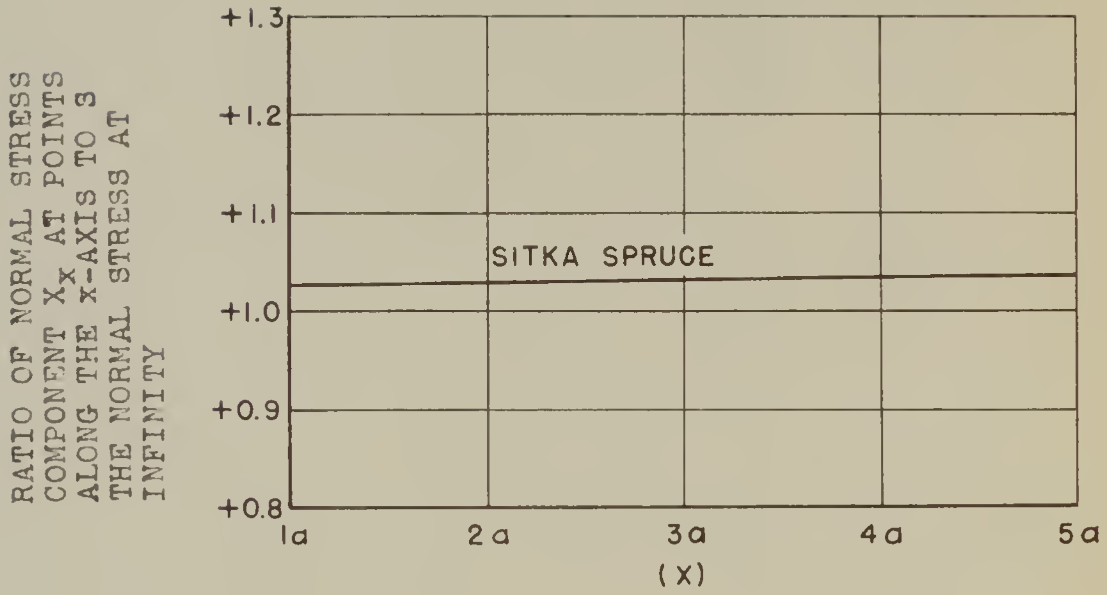

Figure 11.--Variation of the normal stress component X_x at points along the x-axis exterior to a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce.

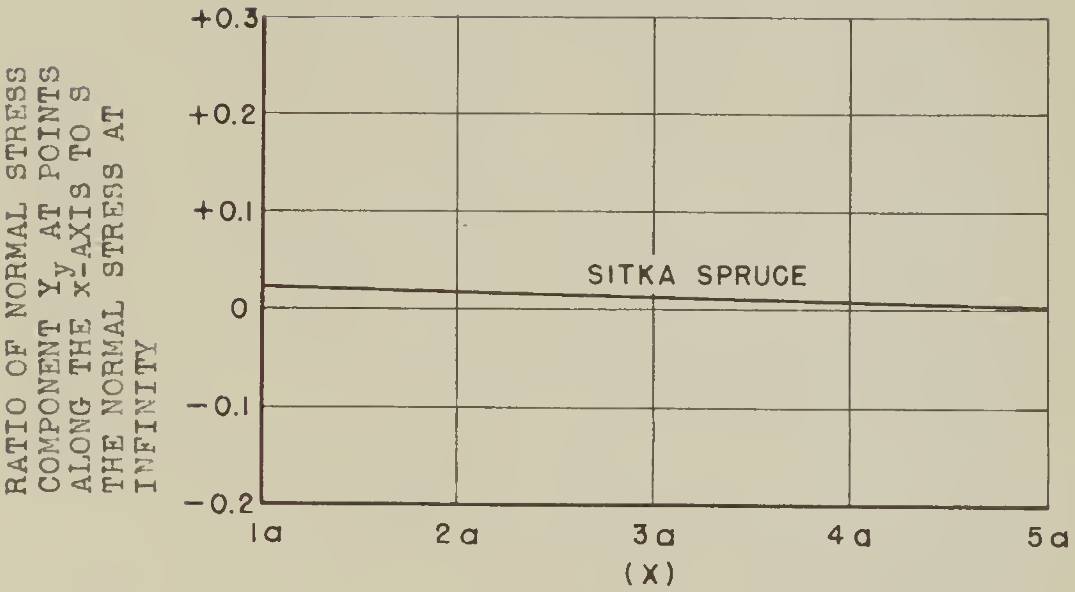

Figure 12.--Variation of the normal stress component Y_y at points along the x-axis exterior to a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce.

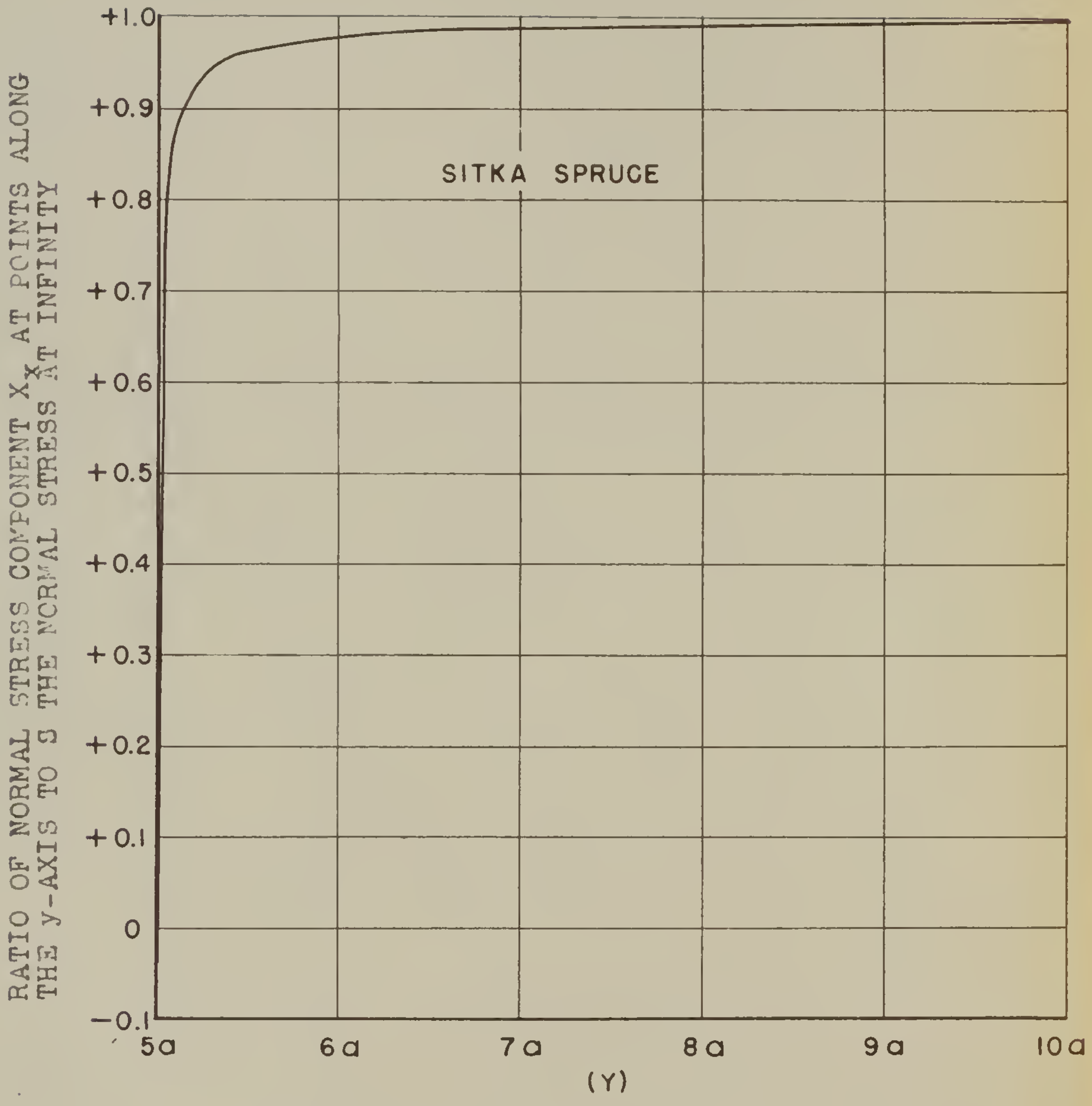

Figure 13.--Variation of the normal stress component X_x at points along the y-axis exterior to a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce.

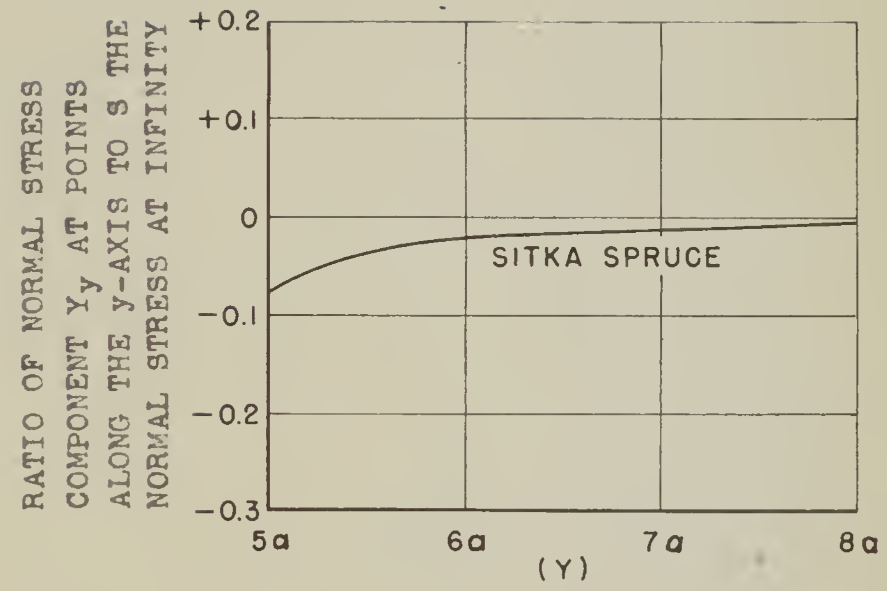

Figure 14.--Variation of the normal stress component X_y at points along the y-axis exterior to a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce.

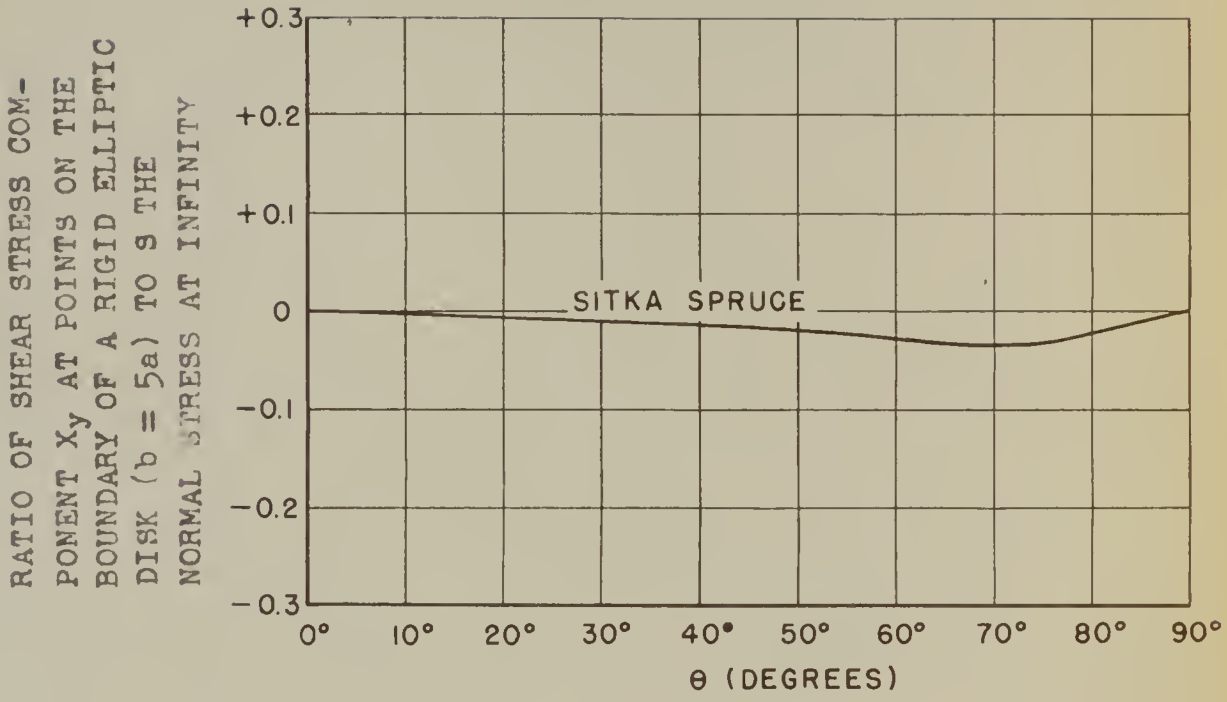

Figure 15.--Variation of the shear stress component X_y at points along the boundary of a rigid elliptic disk (b = 5a) with center at the origin for a plain-sawn plate of Sitka spruce.

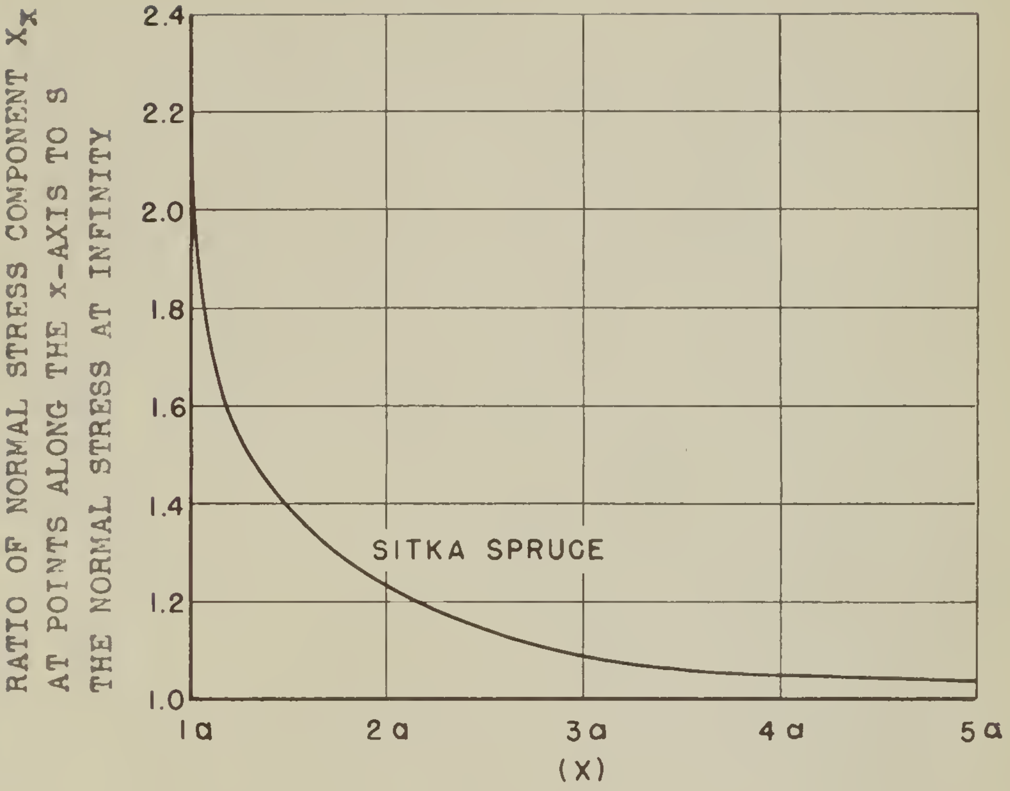

Figure 16.--Variation of the normal stress component X_x at points along the x-axis exterior to a rigid elliptic disk (a = 5b) with center at the origin for a plain-sawn plate of Sitka spruce.

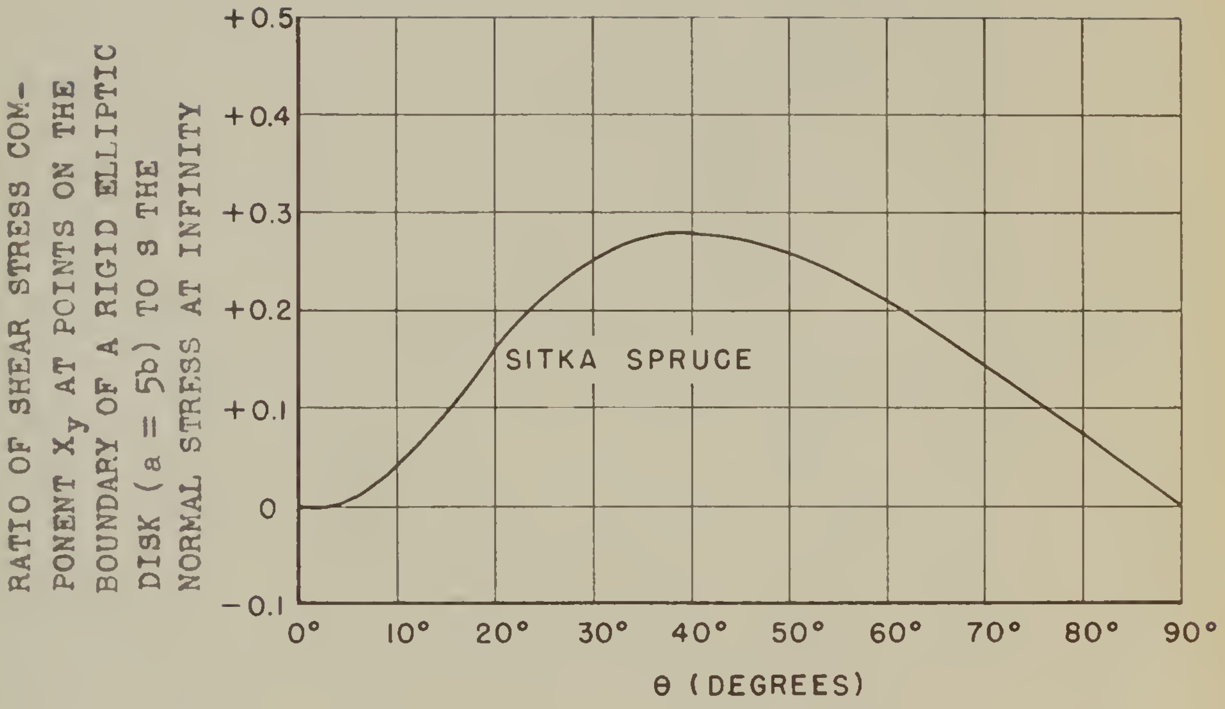

Figure 17.--Variation of the shear stress component X_y at points along the boundary of a rigid elliptic disk (a = 5b) with center at the origin for a plain-sawn plate of Sitka spruce.

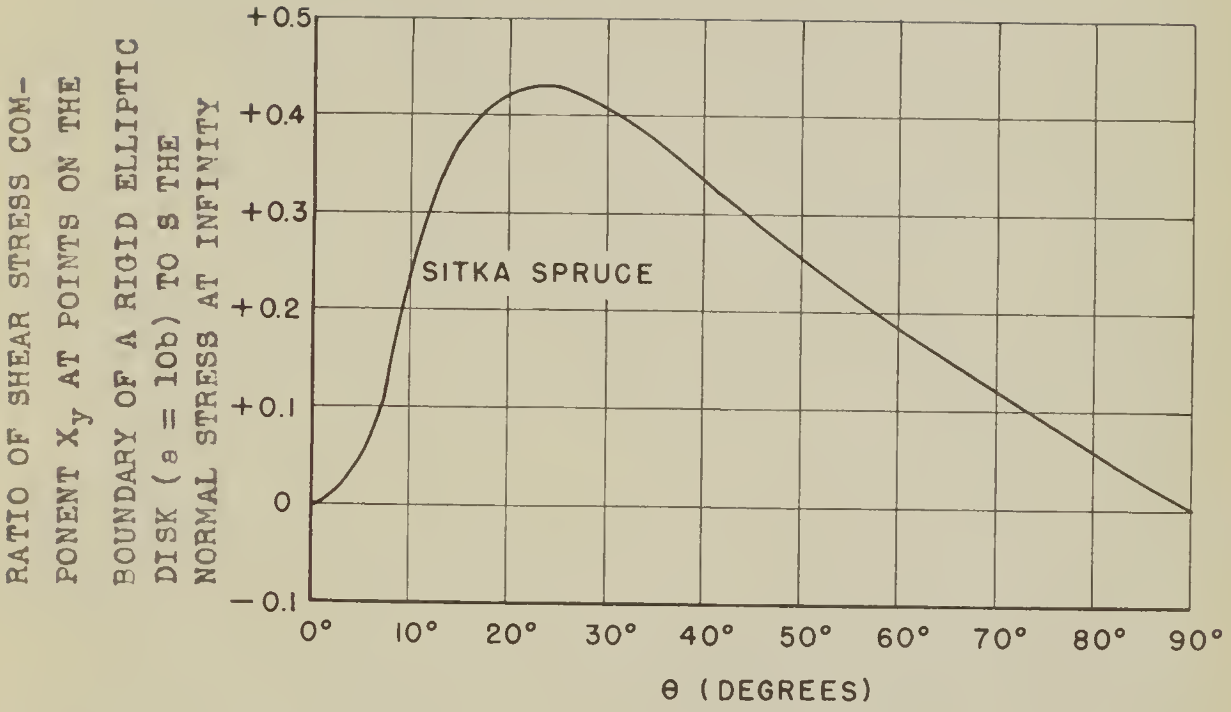

Figure 18.--Variation of the shear stress component X_y at points along the boundary of a rigid elliptic disk (a = 10b) with center at the origin for a plain-sawn plate of Sitka spruce.

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BIOGRAPHICAL SKETCH

The author was born September 26, 1918 at Campbell, Missouri. In 1936 he was graduated from Central High School, Cape Girardeau, Missouri. He was graduated in 1940 from Southeast Missouri State College with the degree of Bachelor of Arts. He attended the banquet given for the ten highest ranking freshmen and also the banquets for the upper five per cent of the Junior and Senior Classes given by the American Association of University Professors. Upon graduation, he received an appointment as a graduate assistant, Department of Physics, State University of Iowa. The degree of Master of Science was conferred in June 1942. He then accepted a position at the Burnside Laboratory of E. I. du Pont de Nemours, Inc. He joined the Navy in 1943, and was graduated from the Naval Aerial Navigation School, Coral Gables, Florida. He was an instructor and later served a year and a half in the Pacific, and was promoted to the rank of lieutenant. After the war, he returned to study at the State University of Iowa. In 1948 he was iniated into the Iowa Chapter of the Society of the Sigma Xi as an associate member. In the fall of 1948, he transferred to the University of Florida. He held a Graduate Fellowship for the year 1949-1950. He is a member of the Mathematical Association of America and the American Mathematical Society.

This dissertation was prepared under the direction of the Chairman of the candidate's Supervisory Committee, and has been approved by all members of the Committee. It was submitted to the Graduate Council and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

T. M. Simpson
Dean

June 5, 1950

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